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The notions of irreducible ideals of the endomorphism ring on the category of rings and the category of modules

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Abstract

Let R commutative ring with multiplicative identity, and M is an R-module. An ideal I of R is irreducible if the intersection of every two ideals of R equals I, then one of them is I itself. Module theory is also known as an irreducible submodule, from the concept of an irreducible ideal in the ring. The set of R − module homomorphisms from M to itself is denoted by $\mathrm{End}_R(M)$. It is called a module endomorphism M of ring R. The set $\mathrm{End}_R(M)$ is also a ring with an addition operation and composition function. This paper showed the sufficient condition of an irreducible ideal on the ring of $\mathrm{End}_R(R)$ and $\mathrm{End}_R(M)$.

INTRODUCTION

A non-empty R set with two binary operations called addition $(+)$ and multiplication $(\cdot)$ which satisfies several axioms, forms an algebraic structure, and it is called a ring denoted by $(R,+,\cdot)$. If $(R,\cdot)$ satisfies the commutative property, it is called a commutative ring. Furthermore, if a ring $(R,\cdot)$ contains a multiplicative identity, it is said to be a ring with a multiplicative identity. This paper assumes $(R,+,\cdot)$ is a commutative ring with multiplicative identity.

In a ring $(R,+,\cdot)$, there are subsets of the ring R, which is a ring over the same operation of $(R,+,\cdot)$ and it is called a subring. A subring has a particular property closed to multiplication operation by elements outside the subring and is known to be ideal. In the ring theory, an irreducible ideal is introduced. An ideal I of ring R is said to be irreducible if for every J, K ideal of ring R with $J \cap K = I$ then $J = I$ or $K = I$ (Mostafanasab and Darani, 2016).

In addition to ring theory, the algebraic structure of a module over a ring is also known. The commutative group $(M,+)$ together with scalar multiplication $\circ : R \times M \to M$, for all $r \in R$ and $m \in M$, holds $\circ(r,m) = r \circ m$ and M satisfies certain axioms then M is called a module over ring R. Furthermore, the ring is not always commutative. The module is divided into the left module and the right module. Module M over ring R is generally a left module over a ring in this paper. The module M over ring R is written as R −module M. In a ring with subrings, the subset of a module also forms a module over the same scalar multiplication on its module, called a submodule.

Suppose I is a left ideal in the ring $(R,+,\cdot)$, then $(I,+)$ also a left module over ring R with the multiplication operation $\cdot$. Let $I = R$ is a trivial ideal. Then ring R can be viewed as a module

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Furthermore, every ideal I in the ring R is a submodule on module R over itself (Wahyuni et al., 2016). It follows from the definition of an irreducible ideal I on ring R, it can also be defined that the irreducible submodule N over R −module M, i.e., for all submodules K and L on R −module M where $K \cap L = N$ which is it holds either $K = N$ or $L = N$ (Abdullah, 2012).

Let M and N be $R-$ modules, a map $f: M \to N$ is said to be module homomorphism if f preserved the addition operation and the scalar multiplication operation on M to N. The set of all $R-$ homomorphisms from M to N is written by $Hom_R(M,N)$. $Hom_R(M,N)$ is a commutative group under the addition function operation. Furthermore, if $M = N$, then $Hom_R(M,M) = End_R(M)$ is also a ring over the addition and the composition function, and it is called module endomorphism M over ring R.Consider R as a module in itself. A homomorphism of the R −module is defined from R to R, denoted by $End_R(R)$. The set $End_R(R)$ is also a ring under the addition and the composition function operation, and it is called an endomorphism ring. There is an isomorphism between ring R and $End_R(R)$.

Based on the explanation, $End_R(M)$ is also a ring over the addition and the composition function. So, the form of the ideal on $End_R(M)$ is investigated. Consider the ring R as a module over itself, so $End_R(R)$ is isomorphic to R (Adkins and Weintraub, 1992). Thus, the ideal on R is also the ideal of $End_R(R)$. This is the special condition of the result of this paper. In this paper, the sufficient condition of an irreducible ideal in $End_R(R)$ and $End_R(M)$ can be investigated.

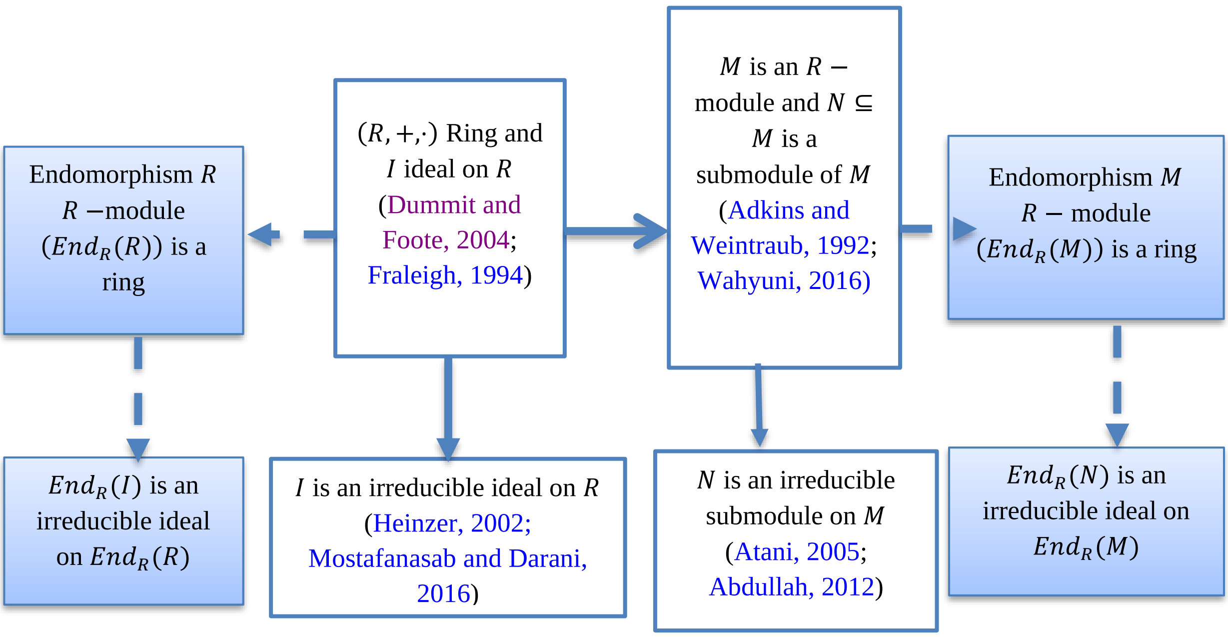

Figure 1. State of the art

The line from Figure 1 shows that other authors had done the research, and the dotted line is the research that worked on in this paper. It follows from Figure 1 that for any ideal I on ring R, the irreducible ideal I on R can be constructed. On the other hand, for any submodule N on R −module M, the irreducible submodule N on R −module M can be constructed. Furthermore, the irreducible ideal on ring endomorphism $End_R(R)$ and $End_R(M)$ can be generalized.

METHODS

The research stage begins with studying ring theory, modules, and the structure of the set of module endomorphisms. Through a search of several works of literature, i.e., (Dummit and Foote, 2004; Fraleigh, 1994; Jimmie and Gilbert, 1984; Malik et al., 2007), and learn about the definition and properties of ring theory. Furthermore, in Holmes (2008) and Mas'oed (2013), the researchers study the ideal concept in a ring. The module structure can be acknowledged in the module homomorphism from M to M' with the concept of endomorphism M over ring R with a multiplicative identity that is an $R-$ homomorphism module over itself (Anderson & Fuller, 1992; Adkins and Weintraub, 1992; Roman, 2008). The concept of endomorphism M over ring R can be learned from Wahyuni et al. (2016). On the other hand, it was also studied that the endomorphism structure M over ring R with the definition of addition and multiplication forms a ring. Furthermore, if $M = R$, then the endomorphism M over ring R can be viewed as an endomorphism R over itself, and it can be shown that endomorphism R over itself is isomorphic with ring R.

The next stage is to study the article of Atani & Atani (2008), Heinzer et al. (2002), Iséki, (1956) and Mostafanasab and Darani (2016), which contains the concept of the ideal and the irreducible ideal in a commutative ring. Then, it can be analyzed the idea of an ideal on ring endomorphism R and the idea of a submodule M that can construct the ideal on module endomorphism M over ring R. The final step is to find a sufficient condition for an ideal in the ring endomorphism R to be an irreducible ideal from the ring endomorphism R in the ring category. Furthermore, acknowledging the concept of the irreducible submodule can use (Albu & Smith, 2009; Abdullah, 2012; Atani, 2005; Khaksari et al., 2006) to construct the irreducible submodule to be an irreducible ideal on module endomorphism M over ring R on module category. The ring and module category concepts can be read in Stenström, (2001) and Wisbauer (1991).

RESULTS AND DISCUSSION

1. The Irreducible of the Endomorphism Ring on the Category of Rings.

A map $\tau: M \to N$ is an $R-$module homomorphism if the function preserves the addition operation from M to N and the scalar multiplication operation on M to N. The set of all $R-$module homomorphisms from M to N is denoted $Hom_R(M, N)$. In the case of $N = M$, it can be written $End_R(M)$ as $R-$module homomorphisms from M to itself and with addition and multiplication defined as function composition and $End_R(M)$ is a ring. Every R ring can be viewed as a module over itself and the set of all $R-$ module homomorphisms from R to itself and denoted $End_R(R)$. Clearly, $End_R(R)$ is a ring with addition and multiplication defined as function composition.

In the ring theory, there is a subring with a particular property closed to multiplication operation by elements outside the subring and is called an ideal. In Adkins and Weintraub (1992), it has been explained that endomorphism ring $End_R(R)$ is isomorphism with ring R. But for any ideal I in R, it is not needed that endomorphism ring R-module I is an ideal of $End_R(R)$. Furthermore, motivated by the characteristics of the ring endomorphism $End_R(R)$, which is a function, the following given a proposition that explains the relationship between the ideal in R and $End_R(R)$.

Proposition 1 *Let R a commutative ring with multiplicative identity and I ideal in R. If $f \in End_R(R)$ and $f(R) \subseteq I$, then $End_R(I)$ is an ideal of $End_R(R)$.*

Proof:

It showed that $End_R(I)$ is an ideal of $End_R(R)$.

1. Let $\alpha, \beta \in End_R(I)$ with $\alpha: I \to I$ and $\beta: I \to I$ is an $R-$module homomorphism, for all $a \in I$ then
$$(\alpha - \beta)(a) = \alpha(a) - \beta(a)$$
Since I ideal, so $\alpha(a) \in I$ dan $\beta(a) \in I$ then $\alpha(a) - \beta(a) \in I$, obtained $(\alpha - \beta)(a) \in I$ and $\alpha - \beta \in End_R(I)$.
2. Let $\alpha \in End_R(I)$ and $f \in End_R(R)$. For all $a \in I$ then
$$(\alpha f)(a) = \alpha\big(f(a)\big)$$
Since $f(R) \subseteq I$, then $f(a) \in I$ it means $\alpha\big(f(a)\big) \in I$ so that $\alpha f \in End_R(I)$.

Hence, from points (1) and (2), it is proved that $End_R(I)$ is an ideal of $End_R(R)$. ■

In the ring theory, it has been discussed the irreducible ideal from the ring. Considering that there is an isomorphism between ring $End_R(R)$ and ring R. Furthermore, irreducible ideal characteristics in ring $End_R(R)$ are identified.

Proposition 2 *Let R a commutative ring with multiplicative identity, I ideal in R and $f(R) \subseteq I$, for all $f \in End_R(R)$. If I irreducible ideal in R with $A \cap B = I$ for all A, B ideal in R, then $End_R(I)$ is an irreducible ideal in $End_R(R)$.*

Proof:

Let I ideal in R and $f(R) \subseteq I$ for all $f \in End_R(R)$. From Proposition 1, then $End_R(I)$ is ideal in $End_R(R)$. It is shown that $End_R(I)$ is an irreducible ideal. It should be known that I is an irreducible ideal, so if $J \cap K = I$ then $J = I$ or $K = I$ for all J, K ideal in R. From the irreducibility of ideal, if $J \cap K = I$ so $End_R(J \cap K) = End_R(I)$ is an ideal in $End_R(R)$ and

1. If $J = I$, then $f(R) \subseteq I = J$. From Proposition 1, then $End_R(J) = End_R(I)$ is an ideal in $End_R(R)$ or
2. If $K = I$, then $f(R) \subseteq I = K$, such that $End_R(K) = End_R(I)$ is an ideal in $End_R(R)$.

Let $End_R(A) \cap End_R(B) = End_R(I)$ ideal with $A, B, I \subseteq R$. To take advantage of the irreducibility of ideal I, first show that $End_R(A) \cap End_R(B) = End_R(A \cap B)$. It is obvious that $End_R(A \cap B) \subseteq End_R(A) \cap End_R(B)$, since $End_R(A) \cap End_R(B) = End_R(I)$ and $f(R) \subseteq I$, for all $f \in End_R(R)$. Conversely, it is shown that $End_R(A) \cap End_R(B) \subseteq End_R(A \cap B)$. Let $f \in End_R(A) \cap End_R(B)$, it means $f \in End_R(A)$ and $f \in End_R(B)$ such that $f(A), f(B), f(A \cap B) \subseteq I$ and $A \cap B = I$ then $f(A \cap B) \subseteq A \cap B$ so that $f \in End_R(A \cap B)$. Hence, $End_R(A) \cap End_R(B) = End_R(A \cap B)$.

Furthermore, it should be noted that $End_R(A) \cap End_R(B) = End_R(I)$. Since $End_R(A) \cap End_R(B) = End_R(A \cap B)$ and $End_R(A \cap B) = End_R(I)$ and the irreducibility of ideal I then $End_R(A) = End_R(I)$ or $End_R(B) = End_R(I)$. So, $End_R(I)$ is an irreducible ideal. ■

To understand the irreducible ideal of $End_R(R)$, the following example is given.

Example 3. Let $\mathbb{Z}$ a commutative ring with multiplicative identity and $5\mathbb{Z}$ an ideal in $\mathbb{Z}$. The set of $5\mathbb{Z}$ is an irreducible ideal of $\mathbb{Z}$, so $End_{\mathbb{Z}}(5\mathbb{Z})$ is an irreducible ideal of $End_{\mathbb{Z}}(\mathbb{Z})$ since for all $End_{\mathbb{Z}}(A), End_{\mathbb{Z}}(B)$ ideal in $End_{\mathbb{Z}}(\mathbb{Z})$ with
$$End_{\mathbb{Z}}(A) \cap End_{\mathbb{Z}}(B) = End_{\mathbb{Z}}(5\mathbb{Z})$$

it means $End_{\mathbb{Z}}(A), End_{\mathbb{Z}}(B)$ should be contained $End_{\mathbb{Z}}(5\mathbb{Z})$.
Ideal in $End_{\mathbb{Z}}(\mathbb{Z})$ which contain $End_{\mathbb{Z}}(5\mathbb{Z})$ is $End_{\mathbb{Z}}(\mathbb{Z})$ and $End_{\mathbb{Z}}(5\mathbb{Z})$.
In this case, the possibility of $End_{\mathbb{Z}}(A)$ and $End_{\mathbb{Z}}(B)$ between the two are:

1. If $End_{\mathbb{Z}}(A) \cap End_{\mathbb{Z}}(B) = End_{\mathbb{Z}}(5\mathbb{Z})$ then $End_{\mathbb{Z}}(A) = End_{\mathbb{Z}}(5\mathbb{Z})$ or $End_{\mathbb{Z}}(B) = End_{\mathbb{Z}}(\mathbb{Z})$.
2. If $End_{\mathbb{Z}}(A) \cap End_{\mathbb{Z}}(B) = End_{\mathbb{Z}}(5\mathbb{Z})$ then $End_{\mathbb{Z}}(A) = End_{\mathbb{Z}}(\mathbb{Z})$ or $End_{\mathbb{Z}}(B) = End_{\mathbb{Z}}(5\mathbb{Z})$.
3. If $End_{\mathbb{Z}}(A) \cap End_{\mathbb{Z}}(B) = End_{\mathbb{Z}}(5\mathbb{Z})$ then $End_{\mathbb{Z}}(A) = End_{\mathbb{Z}}(5\mathbb{Z})$ or $End_{\mathbb{Z}}(B) = End_{\mathbb{Z}}(5\mathbb{Z})$.

From 1 to 3 possibilities above, if $End_{\mathbb{Z}}(A) \cap End_{\mathbb{Z}}(B) = End_{\mathbb{Z}}(5\mathbb{Z})$ then $End_{\mathbb{Z}}(A) = End_{\mathbb{Z}}(5\mathbb{Z})$ or $End_{\mathbb{Z}}(B) = End_{\mathbb{Z}}(5\mathbb{Z})$. It follows that $End_{\mathbb{Z}}(5\mathbb{Z})$ is an irreducible ideal of $End_{\mathbb{Z}}(\mathbb{Z})$.

2. The Irreducible of the Endomorphism Ring on the Category of Modules.

Let $(End_R(M), +, \circ)$ be a ring, based on the previous subsection, $End_R(R)$ is isomorphic to ring R. The irreducible ideal in $End_R(R)$ can be constructed as $End_R(I)$ where I is irreducible ideal in R. In general, for any R −module M constructed as irreducible ideal in $End_R(M)$ by using $N \subseteq M$ submodule in M. First, it is shown that $End_R(N)$ is an ideal in $End_R(M)$ where for all $f \in End_R(M), f(M) \subseteq N$.

Proposition 4 *Let R −module M and $N \subseteq M$ submodule in M. If for all $f \in End_R(M)$, $f(M) \subseteq N$, then $End_R(N)$ is ideal in $End_R(M)$.*

Proof:

If for all $f \in End_R(M)$, $f(M) \subseteq N$, then

1. Let $\alpha, \beta \in End_R(N)$ where $\alpha : N \to N$ and $\beta : N \to N$ homomorphism R −module, $\forall\, x \in N$ holds
$$(\alpha - \beta)(x) = \alpha(x) - \beta(x).$$
Since N submodule in M and $\alpha(x), \beta(x) \in N$ therefore $(\alpha - \beta)(x) \in N$ and $\alpha - \beta \in End_R(N)$.
2. Let $\alpha \in End_R(N)$ and $f \in End_R(M)$, then $\forall\, x \in N$ holds
$$(\alpha f)(x) = \alpha\big(f(x)\big).$$
The assumption $f(M) \subseteq N$ and $f(x) \in N$, then $\alpha\big(f(x)\big) \in N$ and $\alpha f \in End_R(N)$.
Therefore $End_R(N)$ is ideal in $End_R(M)$. ∎

From Proposition 4, the sufficient condition of irreducible ideal $End_R(N)$ in $End_R(M)$ is identified.

Proposition 5 *Let R −module M and $N \subseteq M$ submodule in M where $\alpha(M) \subseteq N$ for all $\alpha \in End_R(M)$. If N is an irreducible submodule in M where $P \cap Q = N$ for all P, Q submodule in M, then $End_R(N)$ is an irreducible ideal in $End_R(M)$.*

Proof:

Let N be a submodule in R −module M and $\alpha(M) \subseteq N$ for all $\alpha \in End_R(M)$. From Proposition 4, it is known $End_R(N)$ as an ideal in $End_R(M)$. It is showed that $End_R(N)$ is an irreducible ideal in $End_R(M)$ i.e., for all $End_R(P), End_R(Q) \subseteq End_R(M)$ ideal where $End_R(P) \cap End_R(Q) = End_R(N)$ holds $End_R(P) = End_R(N)$ or $End_R(Q) = End_R(N)$.

Let $\alpha \in End_R(P)$ and N be an irreducible submodule of M. Then, for $P = N$, $\alpha \in End_R(N)$. Conversely, for $Q = N$, $Q \subseteq P$, $N \subseteq P$, and P is a submodule in R −module M, i.e., $P \subseteq M$, so that $\alpha(P) \subseteq \alpha(M) \subseteq$ N then $\alpha \in End_R(N)$. It follows that $End_R(P) \subseteq End_R(N)$. Conversely, for all $\beta \in End_R(N)$ and N is an irreducible submodule, then for $P = N$, it follows that $\beta \in End_R(P)$. On the other hand, for $Q = N$, $\beta \in End_R(Q)$. From the fact that $P \cap Q = N$ and $Q = N, Q \subseteq P$ then $\beta \in End_R(P)$. The submodule P and $\beta(M) \subseteq P$, then by Proposition 4, it follows that $\beta \in End_R(P)$. Hence, it follows that $End_R(N) \subseteq End_R(P)$.

Let $End_R(P) \cap End_R(Q) = End_R(N)$, it is showed $End_R(P) \cap End_R(Q) = End_R(P \cap Q)$ i.e., $End_R(P) \cap End_R(Q) \subseteq End_R(P \cap Q)$ and $End_R(P \cap Q) \subseteq End_R(P) \cap End_R(Q)$. It is obvious to see that $End_R(P \cap Q) \subseteq End_R(P) \cap End_R(Q)$. Conversely, let $\alpha \in End_R(P) \cap End_R(Q)$ that is $\alpha \in End_R(P)$ and $\alpha \in End_R(Q)$. From the assumption $\alpha(P), \alpha(Q), \alpha(P \cap Q) \subseteq N$, and $P \cap Q = N$, obtained $\alpha(P \cap Q) \subseteq P \cap Q$. Hence, $\alpha \in End_R(P \cap Q)$, so that it follows $End_R(P) \cap End_R(Q) = End_R(P \cap Q)$.

Let $End_R(P) \cap End_R(Q) = End_R(N)$, and it is known that $End_R(P) \cap End_R(Q) = End_R(P \cap Q)$ and $End_R(P \cap Q) = End_R(N)$ so from the property of irreducible submodule N, either $End_R(P) = End_R(N)$ or $End_R(Q) = End_R(N)$. Therefore $End_R(N)$ is an irreducible ideal on $End_R(M)$. ■

From Proposition 5, an example is given.

Example 6 Let $\mathbb{R}-$ module $\mathbb{R}^3$ dan $S = \left\{\begin{bmatrix} a \\ b \\ c \end{bmatrix} | a, b, c \in \mathbb{R}\right\}$ is submodule in $\mathbb{R}^3$. The set S is an irreducible submodule on $\mathbb{R}^3$, and for all $P,\ Q \subseteq \mathbb{R}^3$ submodules where $P \cap Q = S$, it follows that either $P = S$ or $Q = S$. Therefore, $End_{\mathbb{R}}(S)$ is an irreducible ideal on $End_{\mathbb{R}}(\mathbb{R}^3)$, that for all $End_{\mathbb{R}}(P), End_{\mathbb{R}}(Q) \subseteq End_{\mathbb{R}}(\mathbb{R}^3)$ ideal where $End_{\mathbb{R}}(P) \cap End_{\mathbb{R}}(Q) = End_{\mathbb{R}}(S)$ it follows that either $End_{\mathbb{R}}(P) = End_{\mathbb{R}}(S)$ or $End_{\mathbb{R}}(Q) = End_{\mathbb{R}}(S)$.

CONCLUSIONS

Every $End_R(R)$ can be viewed as a ring of the addition operation and the composition function operation. In general, the set $End_R(M)$ is alike. Therefore, an ideal and an irreducible ideal can be constructed on $End_R(R)$ and $End_R(M)$. From the result and discussion, it is found that the sufficient condition for the ideal and irreducible ideal in ring endomorphism on ring category i.e., if I ideal in R and $f \in End_R(R), f(R) \subseteq I$, then $End_R(I)$ is ideal in $End_R(R)$. Furthermore, if I irreducible ideal in R, where $f \in End_R(R)$, $f(R) \subseteq I$, then $End_R(I)$ is an irreducible ideal in $End_R(R)$. In general, the ideal of $End_R(M)$ can be constructed from the submodule N in M so that obtained $End_R(N)$ as an ideal in $End_R(M)$. Then, from the irreducible submodule N in M, obtained $End_R(N)$ as an irreducible ideal on $End_R(M)$. It follows from the result and discussion that it is found that the sufficient condition for the ideal and irreducible ideal in ring endomorphism on module category i.e., if N submodule in M and $f \in End_R(M)$, $f(M) \subseteq N$, then $End_R(N)$ is ideal in $End_R(M)$. Secondly, if N irreducible submodule in M, where $f \in End_R(M)$, $f(M) \subseteq N$, then $End_R(N)$ is an irreducible ideal in $End_R(M)$.

In this research, it is still possible to find the necessary condition for the irreducible ideal on ring endomorphism and the necessary condition for the irreducible ideal from R −module homomorphism. Moreover, researchers also develop this research for the non-unital module over a ring with a non-identity element.

AUTHOR CONTRIBUTIONS STATEMENT

FH investigates the properties of irreducible ideals on rings with unit elements, investigates irreducible ideals in ring endomorphisms, writes articles, looks for references related to research. MF Conducting an investigation of the irreducible ideal of a ring endomorphism of a module on a ring, making a state of the art diagram, completing the discussion of the material and the grammar used. NPP provides direction, guidance and input on the results obtained by authors 1 and 2.

REFERENCES

Abdullah, N. K. (2012). Irreducible Submodules and Strongly Irreducible Submodules. *Tikrit Journal of Pure Science*, *17*(4), 219-224.

Adkins, W. A., and Weintraub, S. H. (1992). *Algebra: An Approach via Module Theory*. New York: Springer-Verlag.

Albu, T., & Smith, P. F. (2009). Primality, Irreducibility, and Complete Irreducibility in Modules Over Commutative Rings. *Rev. Roumaine Math. Pures Appl.*, *54*(4), 275-286.

Anderson, F. W., & Fuller, K. R. (1992). *Rings and Categories of Modules Second Edition*. New York: Springer-Verlag.

Atani, R. E., & Atani, S. E. (2008). Ideal Theory in Commutative Semirings. *Buletinul Academiei De Ştiinţe*, *57*(2), 14-23.

Atani, S. E. (2005). Strongly Irreducible Submodules. *Bulletin of the Korean Mathematical Society*, 121-131.

Dummit, D. S., and Foote, R. M. (2004). *Abstract Algebra Third Edition*. United States of America: John Willey and Sons, Inc.

Fraleigh, J. B. (1994). *A First Course Abstract Algebra*. United States of America: Addison-Wesley Publishing Company, Inc.

Heinzer, W. J., Ratliff, L. J., and Rush, D. E. (2002). Strongly Irreducible Ideals Of A Commutative Ring. *Journal Of Pure and Applied Algebra*, *166*(3), 267-275.

Holmes, R. R. (2008). *Abstract Algebra II*. Alabama, Amerika Serikat: Auburn University.

Iséki, K. (1956). Ideal Theory of Semiring. *Proc. Japan Acad*, *32*(2), 554-559.

Jimmie, G., and Gilbert, L. (1984). *Element Of Modern Algebra*. Boston: PWS-Kent Publishing Company.

Khaksari, A., Ershad, M., & Sharif, H. (2006). Strongly Irreducible Submodules of Modules. *Acta Mathematica Sinica, English Series*, *22*(4), 1189-1196.

Malik, D., Mordeson, J., and Sen, M. (2007). *Introduction to Abstract Algebra*. United States of America: Scientific Word.

Mas'oed, F. (2013). *Struktur Aljabar*. Palembang: Akademia Permata.

Mostafanasab, H., and Darani, A. Y. (2016). 2-Irreducible and Strongly 2-Irreducible Ideals Of Commutative Rings. *Miskolc Mathematical Notes*, *17*(1), 441-455.

Roman, S. (2008). *Advanced Linear Algebra Third Edition*. New York: Springer Science+Business Media.

Stenström, B. (2001). *Lectures on Rings and Modules*. Swedia: Stockholms Universitet.

Wahyuni, S., Wijayanti, I. E., Yuwaningsih, D. A., and Hartanto, A. D. (2016). *Teori Ring dan Modul*. Yogyakarta: Gadjah Mada University Press.

Wisbauer, R. (1991). *Foundations of Module and Ring Theory*. Dusseldorf: Gordon and Breach Science Publishers.