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Extended F-Expansion Method for Solving the modified Korteweg-de Vries (mKdV) Equation

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Abstract
One of the phenomenon in marine science that is often encountered is the phenomenon of water waves. Waves that occur below the surface of seawater are called internal waves. One of the mathematical models that can represent solitary internal waves is the modified Korteweg-de Vries (mKdV) equation. Many methods can be used to construct the solution of the mKdV wave equation, one of which is the extended F-expansion method. The purpose of this study is to determine the solution of the mKdV wave equation using the extended F-expansion method. The result of solving the mKdV wave equation is the exact solutions. The exact solutions of the mKdV wave equation are expressed in the Jacobi elliptic functions, trigonometric functions, and hyperbolic functions. From this research, it is expected to be able to add insight and knowledge about the implementation of the innovative methods for solving wave equations.
Keywords: Extended F-Expansion Method; Modifiead Korteweg-de Vries (mKdV) Equation.

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Introduction

One of the phenomenon in marine science that is often encountered is the phenomenon of water waves. There are water waves that occur at the sea level and some that occur below the surface of seawater. Waves that occur below the surface of seawater are called internal waves. One of the internal waves that are often observed is a solitary wave that has only one peak and propagates by maintaining its shape and speed and there is no backflow (Munk, 1949). This solitary wave motion can be modeled in a mathematical equation to obtain a model approach related to the shape and propagation process towards the coast.

One of the mathematical models that can represent solitary internal waves is the Korteweg-de Vries (KdV) equation. This KdV equation is derived from the basic equation of the ideal fluid, which is incompressible and inviscid. The KdV equation is modified so that the modified Korteweg-de Vries (mKdV) equation is obtained. The mKdV wave equation can be solved using analytic (exact) and numerical (approach) methods.

Many methods have been used by researchers to construct the solution of the mKdV wave equation. Some of them are the F-expansion method (Bashir & Alhakim, 2013), the exp-function method (Chai et al., 2014), the (G’/G)-expansion method (Islam et al., 2015), the inverse scattering transform (Ji & Zhu, 2017), conformable fractional derivative (Nuruddeen, 2018), and the local fractional derivative (Gao et al., 2019) with results in the form of exact solutions. In addition, there are also researchers who use numerical methods such as the Adomian Pade approximation method (Abassy et al., 2004), the numerical inverse scattering (Trogdon et al., 2012), differential quadrature method (Başhan et al., 2016), and a lumped Galerkin method based on cubic B-spline interpolation functions (Ak et al., 2017).

The extended F-expansion method is the development of the F-expansion method by providing additional variables to the solution. This method can be used to solve the problem of nonlinear differential equations in a simple way and produces an exact solutions. This method has been used to solve the modified KdV-KP equation (Al-Fhaid, 2012), the Kudryashov-Sinelshchikov equation (Zhao, 2013), a higher-order wave equation of KdV type (He et al., 2013), the Fourth Order Boussinesq equation (Apriliani, 2015), the Drinfel'd-Sokolov-Wilson (DSW) equation and the Burgers equation (Akbar & Ali, 2017), the Benney-Luke equation and the Phi-4 equation (Islam, Khan, et al., 2017), the MEE circular rod equation and the ZKBBM equation (Islam, Akbar, et al., 2017), nonlinear Klein-Gordon equation (Islam et al., 2018), and the space-time fractional cubic Schrodinger equation (Pandir & Duzgun, 2019).

From the description above, the research related to solving the wave equation is very important to be studied at this time. When the exact solution exists, it can help to understand the dynamic process of the wave equation being modeled. Previous research has not examined the method of extended F-expansion to determine the exact solution of the mKdV wave equation. Therefore, we are interested to study the exact solution of the mKdV wave equation using the extended F-expansion method.

the Research Methods

The data used in this study are secondary data in the form of the modified Korteweg-de Vries (mKdV) equation. The method used is the extended F-expansion method with the main procedure as follows (He et al., 2013):

Step 1

Consider a general nonlinear partial differential equations

$$F(u, u_x, u_t, u_{xx}, u_{xt}, \dots) = 0. \quad (1)$$

Using $u(x,t) = U(\xi)$, $\xi = x - ct$, equation (1) can be written as a nonlinear ordinary differential equation

$$F(U, U', U'', \dots) = 0, \quad (2)$$

Step 2

Suppose the solution of equation (2) can be written as follows:

$$U(\xi) = A_0 + \sum_{i=1}^{n} \left(A_i F^i(\xi) + B_i F^{-i}(\xi) \right), \quad (3)$$

where A_i, B_i $(i = 1,2, \dots n)$ are constants to be determined, n is a positive integer derived from the homogeneous balance principle, and $F(\xi)$ satisfies the following equation:

$$\left(F'(\xi)\right)^2 = h_0 + h_1 F(\xi) + h_2 F^2(\xi) + h_3 F^3(\xi) + h_4 F^4(\xi), \quad (4)$$

where h_0, h_1, h_2, h_3, and h_4 are constants.

Next, the both sides of equation (4) are differentiated to ξ once yield

$$F''(\xi) = \frac{1}{2} h_1 + h_2 F(\xi) + \frac{3}{2} h_3 F^2(\xi) + 2 h_4 F^3(\xi). \quad (5)$$

Step 3

Substituting equations (3), (4), and (5) into equation (2) and setting all the coefficients of $F^j(\xi)$ $(j = 0,1,2,\ldots)$ of the resulting equation to zero yield a set of nonlinear algebraic equation systems for A_0, A_i, and B_i $(i = 1,2,\ldots,n)$.

Step 4

Assuming that the constants A_0, A_i, and B_i $(i = 1,2,\ldots,n)$ can be obtained by solving the algebraic equation systems in step 3 then substituting these constants into equation (3) so that the explicit solutions of equation (1) are obtained which depends on the special conditions chosen for the h_0, h_1, h_2, h_3, and h_4.

the Results of the Research and the Discussion

In this section, the extended F-expansion method is used to determine the exact solution of the mKdV equation (Chai et al., 2014):

$$u_t + u^2u_x + u_{xxx} = 0. \tag{6}$$

Equation (6) is a nonlinear partial differential equation. Based on step 1 of the extended F-expansion method, equation (6) is transformed into a nonlinear ordinary differential equation. The transformations used are $u(x,t) = U(\xi)$, $\xi = x - ct$ so that we obtain

$$-cU' + U^2U' + U''' = 0. \tag{7}$$

Integrating equation (7) once and setting all the integral constants as zero $(k_1 = k_2 = 0)$ so that equation (6) has become a nonlinear ordinary differential equation

$$-3cU + U^3 + 3U'' = 0. \tag{8}$$

From balancing U^3 and U'' in equation (8), we obtain $n = 1$. Based on step 2 of the extended F-expansion method, the solution of equation (8) has the following form

$$u(x,t) = U(\xi) = A_0 + A_1F(\xi) + \frac{B_1}{F(\xi)} \tag{9}$$

where A_0, A_1, and B_1 are constants to be determined and $F(\xi)$ satisfies equations (4) and (5). Based on step 3 of the extended F-expansion method, equations (9), (4), and (5) are substituted into equation (8) yields

$$\begin{aligned} &{A_0}^3 + 6A_0A_1B_1 - 3cA_0 + \frac{3A_1h_1}{2} + \frac{3B_1h_3}{2} + \frac{{B_1}^3+6B_1h_0}{F^3(\xi)} + \frac{3A_0{B_1}^2+\frac{9}{2}B_1h_1}{F^2(\xi)} \\ &+ \frac{3{A_0}^2B_1+3A_1{B_1}^2-3cB_1+3B_1h_2}{F(\xi)} + \left[3{A_0}^2A_1 + 3{A_1}^2B_1 - 3cA_1 + 3A_1h_2\right]F(\xi) \\ &+ \left[3A_0{A_1}^2 + \frac{9}{2}A_1h_3\right]F^2(\xi) + \left[{A_1}^3 + 6A_1h_4\right]F^3(\xi) = 0. \end{aligned} \tag{10}$$

All the coefficients of $F^j(\xi)$ $(j = -3,-2,\ldots,2,3)$ in equation (10) are set to zero so we obtain the following system of nonlinear algebraic equations:

$$\begin{aligned} &F^{-3} : {B_1}^3 + 6B_1h_0 = 0, \\ &F^{-2} : 3A_0{B_1}^2 + \frac{9}{2}B_1h_1 = 0, \\ &F^{-1} : 3{A_0}^2B_1 + 3A_1{B_1}^2 - 3cB_1 + 3B_1h_2 = 0, \end{aligned} \tag{11}$$

$$
\begin{aligned}
F^0 &: {A_0}^3 + 6A_0A_1B_1 - 3cA_0 + \frac{3A_1h_1}{2} + \frac{3B_1h_3}{2} = 0, \\
F &: 3{A_0}^2A_1 + 3{A_1}^2B_1 - 3cA_1 + 3A_1h_2 = 0, \\
F^2 &: 3A_0{A_1}^2 + \frac{9}{2}A_1h_3 = 0, \\
F^3 &: {A_1}^3 + 6A_1h_4 = 0.
\end{aligned}
$$

Based on step 4 of the extended F-expansion method, equation (11) is solved using Maple software with the assumption $h_1 = h_3 = 0$. The results are obtained in Table 1.

Table 1. The Values of A_0, A_1, B_1, c as the Solution of Equation (11)

Case	A_0	A_1	B_1	c
1	0	$\sqrt{-6h_4}$	0	h_2
2	0	$-\sqrt{-6h_4}$	0	h_2
3	0	0	$\sqrt{-6h_0}$	h_2
4	0	0	$-\sqrt{-6h_0}$	h_2
5	0	$\sqrt{-6h_4}$	$\sqrt{-6h_0}$	$h_2 - 6\sqrt{h_0h_4}$
6	0	$\sqrt{-6h_4}$	$-\sqrt{-6h_0}$	$h_2 + 6\sqrt{h_0h_4}$
7	0	$-\sqrt{-6h_4}$	$\sqrt{-6h_0}$	$h_2 + 6\sqrt{h_0h_4}$
8	0	$-\sqrt{-6h_4}$	$-\sqrt{-6h_0}$	$h_2 - 6\sqrt{h_0h_4}$

The values of A_0, A_1, B_1, and c in Table 1 are substituted to equation (9) respectively so that the general solutions of equation (8) are obtained as shown in Table 2.

Table 2. General Solutions of Equation (8)

Case	General Solutions of Equation (8)
1	$u(x,t) = U(\xi) = \sqrt{-6h_4}\,F(\xi)$ with $\xi = x - h_2t$
2	$u(x,t) = U(\xi) = -\sqrt{-6h_4}\,F(\xi)$ with $\xi = x - h_2t$
3	$u(x,t) = U(\xi) = \frac{\sqrt{-6h_0}}{F(\xi)}$ with $\xi = x - h_2t$
4	$u(x,t) = U(\xi) = \frac{-\sqrt{-6h_0}}{F(\xi)}$ with $\xi = x - h_2t$
5	$u(x,t) = U(\xi) = \sqrt{-6h_4}\,F(\xi) + \frac{\sqrt{-6h_0}}{F(\xi)}$ with $\xi = x - \left(h_2 - 6\sqrt{h_0h_4}\right)t$
6	$u(x,t) = U(\xi) = \sqrt{-6h_4}\,F(\xi) - \frac{\sqrt{-6h_0}}{F(\xi)}$ with $\xi = x + \left(h_2 - 6\sqrt{h_0h_4}\right)t$
7	$u(x,t) = U(\xi) = -\sqrt{-6h_4}\,F(\xi) + \frac{\sqrt{-6h_0}}{F(\xi)}$ with $\xi = x + \left(h_2 - 6\sqrt{h_0h_4}\right)t$
8	$u(x,t) = U(\xi) = -\sqrt{-6h_4}\,F(\xi) - \frac{\sqrt{-6h_0}}{F(\xi)}$ with $\xi = x - \left(h_2 - 6\sqrt{h_0h_4}\right)t$

Assuming that $h_1 = h_3 = 0$, then equation (4) becomes

$$\left(F'(\xi)\right)^2 = h_0 + h_2F^2(\xi) + h_4F^4(\xi). \tag{12}$$

The solutions of equation (12) are given in Table 3. Many exact solutions of equation (6) can be obtained by substituting the values of h_0, h_2, h_4 and the function $F(\xi)$ in Table 3 to the general solutions in Table 2.

Table 3. Relations between the Coefficients (h_0, h_2, h_4) and $F(\xi)$ in Equation (12)

Case	h_0	h_2	h_4	$F(\xi)$
1	1	$-(1+m^2)$	m^2	$\mathrm{sn}\xi$
2	1	$-(1+m^2)$	m^2	$\mathrm{cd}\xi$
3	$1-m^2$	$2m^2-1$	$-m^2$	$\mathrm{cn}\xi$
4	m^2-1	$2-m^2$	-1	$\mathrm{dn}\xi$
5	m^2	$-(1+m^2)$	1	$\mathrm{ns}\xi$
6	m^2	$-(1+m^2)$	1	$\mathrm{dc}\xi$
7	$-m^2$	$2m^2-1$	$1-m^2$	$\mathrm{nc}\xi$
8	-1	$2-m^2$	m^2-1	$\mathrm{nd}\xi$
9	1	$2-m^2$	$1-m^2$	$\mathrm{sc}\xi$
10	1	$2m^2-1$	$-m^2(1-m^2)$	$\mathrm{sd}\xi$
11	$1-m^2$	$2-m^2$	1	$\mathrm{cs}\xi$
12	$-m^2(1-m^2)$	$2m^2-1$	1	$\mathrm{ds}\xi$

Table 4. Jacobi Elliptic Functions Degenerate into Hyperbolic Functions when $m \to 1$

$\mathrm{sn}(\xi) \to \tanh(\xi)$	$\mathrm{cn}(\xi) \to \mathrm{sech}(\xi)$	$\mathrm{dn}(\xi) \to \mathrm{sech}(\xi)$	$\mathrm{sc}(\xi) \to \sinh(\xi)$
$\mathrm{sd}(\xi) \to \sinh(\xi)$	$\mathrm{cd}(\xi) \to 1$	$\mathrm{ns}(\xi) \to \coth(\xi)$	$\mathrm{nc}(\xi) \to \cosh(\xi)$
$\mathrm{nd}(\xi) \to \cosh(\xi)$	$\mathrm{cs}(\xi) \to \mathrm{csch}(\xi)$	$\mathrm{ds}(\xi) \to \mathrm{csch}(\xi)$	$\mathrm{dc}(\xi) \to 1$

Table 5. Jacobi Elliptic Functions Degenerate into Trigonometric Functions when $m \to 0$

$\mathrm{sn}(\xi) \to \sin(\xi)$	$\mathrm{cn}(\xi) \to \cos(\xi)$	$\mathrm{dn}(\xi) \to 1$	$\mathrm{sc}(\xi) \to \tan(\xi)$
$\mathrm{sd}(\xi) \to \sin(\xi)$	$\mathrm{cd}(\xi) \to \cos(\xi)$	$\mathrm{ns}(\xi) \to \csc(\xi)$	$\mathrm{nc}(\xi) \to \sec(\xi)$
$\mathrm{nd}(\xi) \to 1$	$\mathrm{cs}(\xi) \to \cot(\xi)$	$\mathrm{ds}(\xi) \to \csc(\xi)$	$\mathrm{dc}(\xi) \to \sec(\xi)$

Each case in Table 3 is substituted to the general solutions in Table 2 so we obtain the exact solutions of the mKdV equation, which is equation (6) in the form of Jacobi elliptic functions. In addition, the Jacobi elliptic functions can degenerate when $m \to 1$ become hyperbolic functions using Table 4 and degenerate when $m \to 0$ become trigonometric functions using Table 5.

The exact solutions of the mKdV equation in the form of Jacobi elliptic functions, hyperbolic functions, and trigonometric functions for Table 3 case 1 are obtained in the Table 6, Table 7, and Table 8.

Table 6. Exact Solutions of Equation (6) in Jacobi Elliptic Functions for Table 3 Case 1

Case	Exact Solutions in Jacobi Elliptic Functions
1	$u(x,t) = U(\xi) = \sqrt{-6m^2}\,\mathrm{sn}\xi$ with $\xi = x + (1+m^2)t$
2	$u(x,t) = U(\xi) = -\sqrt{-6m^2}\,\mathrm{sn}\xi$ with $\xi = x + (1+m^2)t$
3	$u(x,t) = U(\xi) = \frac{\sqrt{-6}}{\mathrm{sn}\xi}$ with $\xi = x + (1+m^2)t$
4	$u(x,t) = U(\xi) = \frac{-\sqrt{-6}}{\mathrm{sn}\xi}$ with $\xi = x + (1+m^2)t$
5	$u(x,t) = U(\xi) = \sqrt{-6m^2}\,\mathrm{sn}\xi + \frac{\sqrt{-6}}{\mathrm{sn}\xi}$ with $\xi = x + \left((1+m^2) + 6\sqrt{m^2}\right)t$
6	$u(x,t) = U(\xi) = \sqrt{-6m^2}\,\mathrm{sn}\xi - \frac{\sqrt{-6}}{\mathrm{sn}\xi}$ with $\xi = x - \left((1+m^2) + 6\sqrt{m^2}\right)t$
7	$u(x,t) = U(\xi) = -\sqrt{-6m^2}\,\mathrm{sn}\xi + \frac{\sqrt{-6}}{\mathrm{sn}\xi}$ with $\xi = x - \left((1+m^2) + 6\sqrt{m^2}\right)t$
8	$u(x,t) = U(\xi) = -\sqrt{-6m^2}\,\mathrm{sn}\xi - \frac{\sqrt{-6}}{\mathrm{sn}\xi}$ with $\xi = x + \left((1+m^2) + 6\sqrt{m^2}\right)t$

Table 7. Exact Solutions of Equation (6) in Hyperbolic Functions for Table 3 Case 1

Case	Exact Solutions in Hyperbolic Functions
1	$u(x,t) = U(\xi) = \sqrt{-6}\tanh(\xi)$ with $\xi = x + 2t$
2	$u(x,t) = U(\xi) = -\sqrt{-6}\tanh(\xi)$ with $\xi = x + 2t$
3	$u(x,t) = U(\xi) = \frac{\sqrt{-6}}{\tanh(\xi)}$ with $\xi = x + 2t$
4	$u(x,t) = U(\xi) = \frac{-\sqrt{-6}}{\tanh(\xi)}$ with $\xi = x + 2t$
5	$u(x,t) = U(\xi) = \sqrt{-6}\tanh(\xi) + \frac{\sqrt{-6}}{\tanh(\xi)}$ with $\xi = x + 8t$
6	$u(x,t) = U(\xi) = \sqrt{-6}\tanh(\xi) - \frac{\sqrt{-6}}{\tanh(\xi)}$ with $\xi = x - 8t$
7	$u(x,t) = U(\xi) = -\sqrt{-6}\tanh(\xi) + \frac{\sqrt{-6}}{\tanh(\xi)}$ with $\xi = x - 8t$
8	$u(x,t) = U(\xi) = -\sqrt{-6}\tanh(\xi) - \frac{\sqrt{-6}}{\tanh(\xi)}$ with $\xi = x + 8t$

Table 8. Exact Solutions of Equation (6) in Trigonometric Functions for Table 3 Case 1

Case	Exact Solutions in Trigonometric Functions
1,2	$u(x,t) = U(\xi) = 0$ with $\xi = x + t$
3,5	$u(x,t) = U(\xi) = \frac{\sqrt{-6}}{\sin(\xi)}$ with $\xi = x + t$
4,8	$u(x,t) = U(\xi) = \frac{-\sqrt{-6}}{\sin(\xi)}$ with $\xi = x + t$
6	$u(x,t) = U(\xi) = \frac{-\sqrt{-6}}{\sin(\xi)}$ with $\xi = x - t$
7	$u(x,t) = U(\xi) = \frac{\sqrt{-6}}{\sin(\xi)}$ with $\xi = x - t$

Based on Table 6, Table 7, and Table 8, there are 8 solutions in the form of Jacobi elliptic functions, 8 solutions in the form of hyperbolic functions, and 4 solutions in the form of trigonometric functions because there are cases that obtain the same solutions and there are solutions in the form of constant functions. In this article, we only discuss one case in Table 3 as an illustration, while other cases can be solved in a similar way.

The mKdV wave equation studied in this paper has solved in previous studies using the F-expansion method (Bashir & Alhakim, 2013). In this paper, we use the extended F-expansion method to determine the exact solution of the mKdV wave equation. The general form of solutions obtained using the extended F-expansion method are more complete than using the F-expansion method. The result of an exact solutions are expressed in the form of Jacobi elliptic functions, hyperbolic functions, and trigonometric functions with some of the solutions are the same as previous studies but there are also several different variations. In addition, several other methods have also been used by previous researchers to construct the exact solutions of the mKdV wave equation (Chai et al., 2014; Islam et al., 2015; Ji & Zhu, 2017; Nuruddeen, 2018; Gao et al., 2019). However, the exact solutions obtained in previous studies were expressed in the form of functions that were different from this study.

Conclusion and Suggestion

The extended F-expansion method is one of the most effective methods in determining the exact solutions of various differential equations. In this study, the modified Korteweg-de Vries (mKdV) equation was successfully solved using the extended F-expansion method and some exact solutions are expressed in the form of Jacobi elliptic functions, hyperbolic functions, and trigonometric functions. The correctness of all the exact solutions is verified by substituting the solutions into original equation (mKdV). The exact solution obtained using the extended F-expansion method is more varied than the exact solution obtained in previous studies. In this study, the extended F-expansion method is used to determine the exact solutions of the third-order mKdV equation. Further studies are needed to apply this method in determining the solution of the mKdV equation with a higher-order.

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