General Formula for Limit of Square Function at Infinity

Bonno Andri Wibowo 1 *, Ikhsan Maulidi 2, Windiani Erliana 3

1,3 Institut Pertanian Bogor, Jalan Raya Dramaga, Bogor 16680, Indonesia.
2 Universitas Syiah Kuala, Jalan Teuku Nyak Arief Darussalam, Banda Aceh 23111, Indonesia.
* Corresponding Author. E-mail: bonno1818@gmail.com

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Abstract

Determination of the limit value of a function is an important things. Basically, the limit is used to look at the "properties" function value around the point. In this paper, we provide the general formula for the limit of square root function at infinite. This general formula comes from the development of a commonly known base formula. We use some simple algebra theorems to develop it. The result is very similar to the basic formula for limit of square root function at infinite.

Keywords: General Formula, Limit at Infinity, Square Function.

INTRODUCTION

Determination of the limit value of a function is an important aspect not only for calculus but also for other applied mathematics. Basically, the limit is used to look at the "properties" function value around the point. Limit at infinite is often used to see the convergence of a function. A function is said to be convergent if \( \lim_{x \to \infty} f(x) = L < \infty \). One of research article about limit at infinity is about behaviour analysis for the solution of a problem (Kaewkhao, A. & Intep, 2013). One particular form of square root function which is often discussed on popular calculus books is

\[
\lim_{x \to \infty} \sqrt{ax^2 + bx + c} - \sqrt{px^2 + qx + r}
\]

\[
= \begin{cases}
\frac{b-q}{2\sqrt{p}} ; a = p \\
\infty ; a > p \\
-\infty ; a < p
\end{cases}
\]  

(Steward, 2011).

The Proof is omitted.

This study is a development of (1) to obtain the general formula of the square root function limit at infinite. Proof of general formula of the function by using algebraic properties of series and limit. Development is does gradually to see the pattern of proof.

Algebraic properties of series and limit which be used are

\[
\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i
\]

\[
\sum_{i=1}^{n} a_i p = p \sum_{i=1}^{n} a_i
\]

\[
\lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n
\]

Proof. (Bartle, R.G. & Shebert, 2010)
**METHODS**

This research is a literature review and the development of basic concepts of limit functions. Firstly, we provide some technical Lemmas which are useful for proving the main Theorem. Then the main theorem was given and we give some examples for result of this article.

**Technical Lemmas**

In this section, we provide some lemmas to be used in proving of the Theorem

**Lemma 1:**

Let $a, b, c, p_j, q_j, r_j \in R$ and $a_p j > 0$.

If $\sqrt[n]{a} = \sum_{j=1}^{n} p_j^j$ then

$$\lim_{x \to \infty} (\sqrt[n]{a} x^2 + bx + c - \sum_{j=1}^{n} p_j x^2 + q_j x + r_j) = \frac{b}{2\sqrt[n]{a}} - \sum_{j=1}^{n} \frac{q_j}{2\sqrt[n]{p_j}}.$$  

**Proof:**

$$\lim_{x \to \infty} (\sqrt[n]{a} x^2 + bx + c - \sum_{j=1}^{n} p_j x^2 + q_j x + r_j) = \lim_{x \to \infty} (\sqrt[n]{a} x^2 + bx + c - \sum_{j=1}^{n} p_j x^2 + q_j x + r_j)$$

$$= \lim_{x \to \infty} \left( \frac{p_j b x^2 + p_j b x + p_j c}{a} - \frac{q_j}{a} \right)$$

$$= \sum_{j=1}^{n} \lim_{x \to \infty} \left( \frac{p_j b x^2 + p_j b x + p_j c}{a} - \frac{q_j}{a} \right)$$

$$= \sum_{j=1}^{n} \frac{p_j b x^2 + p_j b x + p_j c}{a} - \sum_{j=1}^{n} \frac{q_j}{a}$$

$$= \sum_{j=1}^{n} \frac{p_j b x^2 + p_j b x + p_j c}{a} - \sum_{j=1}^{n} \frac{q_j}{a}$$

This completes the proof of Lemma 1.

**RESULTS AND DISCUSSION**

**Theorem**

Let $a_i, b_i, c_i, p_j, q_j, r_j, b, c \in R$ and $a_p j > 0$ also let $b_i = ba_i$ and $c_i = ca_i$.  

Then

$$\lim_{x \to \infty} (\sum_{i=1}^{m} a_i x^2 + b_i x + c_i - \sum_{j=1}^{n} p_j x^2 + q_j x + r_j) = \sum_{i=1}^{m} \frac{b_i}{2\sqrt[i]{a_i}} - \sum_{j=1}^{n} \frac{q_j}{2\sqrt[j]{p_j}}.$$  

(4)

**Proof:**

**Case 1:** $\sum_{i=1}^{m} \sqrt[i]{a_i} = \sum_{j=1}^{n} \sqrt[j]{p_j}$

$$\lim_{x \to \infty} \left( \sum_{i=1}^{m} a_i x^2 + b_i x + c_i - \sum_{j=1}^{n} p_j x^2 + q_j x + r_j \right)$$

$$= \lim_{x \to \infty} \left( \sum_{i=1}^{m} a_i x^2 + b_i x + c_i - \sum_{j=1}^{n} p_j x^2 + q_j x + r_j \right)$$

Substitute (2) to (5)

$$= \sum_{i=1}^{m} \frac{b_i}{2\sqrt[i]{a_i}} - \sum_{j=1}^{n} \frac{q_j}{2\sqrt[j]{p_j}}$$

From (3) we have
Case 2: \( \frac{m}{n} = \frac{a}{b} > \frac{n}{m} = \frac{p}{j} \) if \( a_i \neq 0 \) and \( b_i \neq 0 \), then

Let: \( \sum_{i=1}^{m} a_i^2 = \sum_{j=1}^{n} b_j^2 + k \) with \( k > 0 \)

\[
\lim_{x \to \infty} \left( \sum_{i=1}^{m} a_i^2 x^2 + b_i x + c_i \right) = \lim_{x \to \infty} \left( \sum_{j=1}^{n} b_j^2 x^2 + q_j x + r_j \right) = \lim_{x \to \infty} \left( \sum_{i=1}^{m} a_i^2 x^2 + a_i b_i x + a_i c_i \right)
\]

\[
\lim_{x \to \infty} \left( \sum_{j=1}^{n} b_j^2 x^2 + q_j x + r_j \right) = \lim_{x \to \infty} \left( \sum_{j=1}^{n} b_j^2 x^2 + q_j x + r_j \right) = \lim_{x \to \infty} \left( \sum_{j=1}^{n} b_j^2 x^2 + q_j x + r_j \right) = \lim_{x \to \infty} \left( \sum_{j=1}^{n} b_j^2 x^2 + q_j x + r_j \right)
\]

This completes the proof of Theorem.

Appendix

In this section, the authors provide examples of the use of the obtained formula and compare the results with Laurent series approach.

Example of Lemma 1

\[
\lim_{x \to \infty} (\sqrt{4x^2 + x - 1} - \sqrt{2x^2 - x + 1}) = \frac{1}{\sqrt{2}} - \frac{-1}{2\sqrt{2}} = \frac{5}{4}
\]

Laurent series approach:

\[
\lim_{x \to \infty} \frac{4x^2 + x - 1}{\sqrt{x^2} - x + 1 - \sqrt{2x^2 - 2x + 3}} = \frac{7}{4} - \frac{105}{64} - \frac{591}{512} x + O\left(\left(\frac{1}{x}\right)^3\right).
\]

So

\[
\lim_{x \to \infty} \frac{4x^2 + x - 1}{\sqrt{x^2} - x + 1 - \sqrt{2x^2 - 2x + 3}} = \frac{7}{4} - \frac{105}{64} - \frac{591}{512} x + O\left(\left(\frac{1}{x}\right)^3\right)
\]

Example of Lemma 2/Theorem

\[
\lim_{x \to \infty} \frac{\sqrt{x^2 + 2x - 1 + \sqrt{64x^2 + 128x - 64}}}{\sqrt{x^2 + 3x + 3 + \sqrt{9x^2 + 5x + 10 + \sqrt{16x^2 + 8x - 3}}}} = \frac{2}{\sqrt{2}} + \frac{128}{2\sqrt{2}} - \frac{124416}{512} x + \frac{1129361}{124416} x + O\left(\left(\frac{1}{x}\right)^3\right).
\]

Laurent series approach:

\[
\lim_{x \to \infty} \frac{\sqrt{x^2 + 2x - 1 + \sqrt{64x^2 + 128x - 64}}}{\sqrt{x^2 + 3x + 3 + \sqrt{9x^2 + 5x + 10 + \sqrt{16x^2 + 8x - 3}}}} = \frac{2}{\sqrt{2}} + \frac{128}{2\sqrt{2}} - \frac{124416}{512} x + \frac{1129361}{124416} x + O\left(\left(\frac{1}{x}\right)^3\right)
\]

Other Example (When \( b_i \) and \( c_i \) values which do not depend on \( a_i \) values)

\[
\lim_{x \to \infty} \frac{\sqrt{x^2 + 2x - 1 + \sqrt{64x^2 + 128x - 64}}}{\sqrt{x^2 + 3x + 3 + \sqrt{9x^2 + 5x + 10 + \sqrt{16x^2 + 8x - 3}}}} = \frac{2}{\sqrt{2}} + \frac{16}{2\sqrt{2}} - \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}} - \frac{8}{2\sqrt{2}} = \frac{11}{12}
\]

Laurent series approach:

\[
\lim_{x \to \infty} \frac{\sqrt{x^2 + 2x - 1 + \sqrt{64x^2 + 128x - 64}}}{\sqrt{x^2 + 3x + 3 + \sqrt{9x^2 + 5x + 10 + \sqrt{16x^2 + 8x - 3}}}} = \frac{2}{\sqrt{2}} + \frac{16}{2\sqrt{2}} - \frac{3}{2\sqrt{2}} - \frac{5}{2\sqrt{2}} - \frac{8}{2\sqrt{2}} = \frac{11}{12}
\]

CONCLUSION AND SUGGESTION

The general formula of the square root function limit at infinity in (4) is very similar with the basic formula of square root function limit in (1). The author strongly
believes that although $b_i$, and $c_i$ are independent of $a_i$ values, we still can use this general formula. In the appendix section, the author gives examples of limit function and compare the results of calculations with the calculation of the limit value using Laurent series approach method (Rodriguez, R.E, Kra, I., & Gilman, 2012).

Further researchs that can be done is to provide a mathematical proof for $b_i$ and $c_i$ values which do not depend on $a_i$ values or giving a counter example for the general formula.

REFERENCES