The relationship of the formulas for the number of connected vertices labeled graphs with order five and order six without loops

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ABSTRACT
Given a graph with n points and m lines. If each vertex is labeled, then it can be constructed many graphs, connected, or disconnected graphs. A graph G is called a connected graph if there is at least one path that connects a pair of vertices in G. In addition, the graph formed may be simple or not simple. A simple graph is a graph that does not contain loops or parallel lines. A loop is a line that connects a point to itself, and a parallel line is two or more lines that connect the same pair of points. This paper will discuss the relationship between the formula patterns for calculating the number of connected graphs labeled with vertices of order five and six without loops.

INTRODUCTION
Many real-life problems can be represented using graph theory. Graph theory is one of the fields in mathematics that is widely used as a tool to represent problems. Many branches of science that use the concept of graph theory, including chemistry, biology, computer science, economics, engineering, and others. In the field of chemistry, for example, Burch (2018) used graphs to model chemical compounds to provide an overview of the physical properties of these chemical compounds; in the field of biology Mathur & Adlakha (2016) used a combination of trees to represent DNA, Husu & Bryant (2006) and Brandes & Cornelsen (2009) used a labeled tree to represent the evolution of taxa called phylogenetic trees; in the field of computer science Hsu & Lin (2008) comprehensively discussed graph concepts related to network design, and Etawi (2014) used the minimum spanning tree concept, complete graph, and cycle graph to generate a ciphertext and others.

A Graph G is defined as an unordered pair of (V(G), E(G)) where (G) = {v1, v2, ..., vn} is a set of points/vertices, (G), and (G) = {e1, e2, ..., en} is the set of edges or lines
of $G$. An edge or line with the same starting point and ending point is called a loop, while a parallel line is two or more lines connecting the same points. A simple graph is a graph that does not contain loops or parallel lines, otherwise, the graph is called not simple. Vertices on a graph are generally used to represent cities/stations/airports, and so on, while lines are used to represent roads/railroads/airlines, and so on. For example, in a transportation problem, cities can be represented by vertices while roads connecting cities can be represented by lines. In addition, each line can be assigned a value that can represent non-structural information such as distance, cost, time, and others. Thus, by representing the problem in the form of a graph, the problems encountered can be easily visualized.

Graph enumeration was pioneered by Cayley (1874) when calculating the number of $C_n H_{2n+2}$ hydrocarbon isomers which turned out to be related to counting rooted trees in graphs. Harary & Palmer (1973) provided a technique for calculating the number of graphs in general. Wamiliana et al. (2017) conducted research to determine the number of disconnected vertices labeled graphs of order five without parallel lines. Amanto et al. (2017) provided a formula to determine the number of disconnected vertices labeled graphs with a maximum order of four. Furthermore, Amanto et al. (2018) provided a formula to calculate disconnected vertex labeled graph of order five with a maximum of six parallel edges without loops. In the same year, Dracjat et al. (2018) conducted research to calculate the number of connected graphs labeled with vertices of order five with a maximum of two parallel lines or loops and a maximum of six non-parallel lines. Wamiliana et al. (2019) investigated a formula to calculate the number of vertices labeled graphs of order five with a maximum of five parallel lines and no loops. Puri et al. (2021) provided a formula to calculate the number of connected graphs labeled with vertices of order 6 with a maximum of 30 lines without loops, and Putri et al. (2021) conducted research to determine the number of disconnected vertices labeled graphs of order 6 with a maximum of 20 parallel lines and no loops.

In this research, the patterns of the formula for the number of connected vertex labels of order five and six without loops will be observed to find the relationship between the formula.

**METHOD**

Given a graph with $n$ vertices and $m$ lines with each vertex is labeled. Then, many graphs can be constructed, whether connected or disconnected, simple or not simple. In the construction process, isomorphic graphs are only counted as one graph. Two graphs $G$ and $G'$ are said to be equivalent (or sometimes are called isomorphic) if there is a one-one correspondence between the vertices of the two graphs and between their lines so that if the line $e$ is adjacent to the points $u$ and $v$ in $G$ then the line $e'$ in $G'$ also adjacent to the points $u'$ and $v'$ in $G'$.

**Notation:**

$n$ = the number of vertices

$m$ = the number of edges

$g$ = the number of non-parallel edges

$g = m - \sum_{i=1}^{\infty} p_i$

$p_i$ = the number of $i$ parallel edges; $i \geq 2$

$j_i$ = coefficient for the number of $i$ - parallel edges

$t$ = the number of edges that connect the same pairs of vertices (parallel edges are counted as one).

$m = \sum_{i=2}^{\infty} j_i p_i + g$
Figure 1. An Example of a Graph of Order Six with \( m = 8, g = 3, p_2=1, p_3=1, j_2=1, \) and \( j_3=1. \)

Based on Figure 1, the total numbers of edges \( m = 8, \) the number of non-parallel edges \( g \) is 3, the number of two parallel edges \( p_2 \) is 1 \( (j_2=1), \) and the number of three parallel edges \( p_3 \) is also 1 \( (j_3=1). \)

\[
m = j_2p_2 + j_3p_3 + g \\
= (1)(2) + (1)(3) + 3 \\
= 8
\]

To find the formula for the number of connected vertices labeled graph of order six without loops, first, constructed the patterns of the graph by comparing it with the pattern of order five and then find which patterns do not exist in order five. Next, calculate the number of graphs obtained in every pattern and then predict and determine the formula by comparing it with the formula of order five.

Table 1. Comparison of Some Patterns of Connected Vertex Labeled Graphs of Order Five and Six without Loops
RESULTS AND DISCUSSION

From the observation, it is found that after grouping the graphs obtained by using \( t \) and \( m \), the number of graphs is described in Table 2.

Table 2. Comparison of the Number of Connected Vertices Labeled Graphs of Order Five and Six without Loops.

<table>
<thead>
<tr>
<th>( m ) ( n=5 )</th>
<th>( t=5 )</th>
<th>( n=6 )</th>
<th>( t=6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1x222</td>
<td>1x1296</td>
<td>1x205</td>
</tr>
<tr>
<td>6</td>
<td>5x222</td>
<td>5x1296</td>
<td>1x205</td>
</tr>
<tr>
<td>7</td>
<td>15x222</td>
<td>15x1296</td>
<td>6x205</td>
</tr>
<tr>
<td>8</td>
<td>35x222</td>
<td>35x1296</td>
<td>21x205</td>
</tr>
<tr>
<td>9</td>
<td>70x222</td>
<td>70x1296</td>
<td>56x205</td>
</tr>
<tr>
<td>10</td>
<td>126x222</td>
<td>126x1296</td>
<td>126x205</td>
</tr>
<tr>
<td>11</td>
<td>210x1296</td>
<td>252x1980</td>
<td>252x1980</td>
</tr>
<tr>
<td>12</td>
<td>330x1296</td>
<td>462x1980</td>
<td>462x1980</td>
</tr>
<tr>
<td>13</td>
<td>495x1296</td>
<td>792x1980</td>
<td>792x1980</td>
</tr>
<tr>
<td>14</td>
<td>715x1296</td>
<td>1287x1980</td>
<td>1287x1980</td>
</tr>
</tbody>
</table>

The authors provide the results for the number of connected vertices labeled...
Theorem 1:

Given \( n=5 \), \( 4 \leq m \leq 10 \), and \( t=5 \), the number of vertices labeled connected graphs of order 5 without loop is \( N(G_{n,m,t}) = 222 \times C_{4}^{(m-1)} \).

Proof:
See the sequence of numbers above. The fixed difference occurs in the fourth level, therefore the polynomial that is related to that sequence is a polynomial of order four \( P_{4}(m) = \alpha_{4}m^{4} + \alpha_{3}m^{3} + \alpha_{2}m^{2} + \alpha_{1}m + \alpha_{0} \).

Thus, we get the following system of the equation:

\[
\begin{align*}
222 &= 625\alpha_{4} + 125\alpha_{3} + 25\alpha_{2} + 5\alpha_{1} + \alpha_{0} \quad (1) \\
1110 &= 1296\alpha_{4} + 216\alpha_{3} + 36\alpha_{2} + 6\alpha_{1} + \alpha_{0} \quad (2) \\
3330 &= 2401\alpha_{4} + 343\alpha_{3} + 49\alpha_{2} + 7\alpha_{1} + \alpha_{0} \quad (3) \\
7770 &= 4096\alpha_{4} + 512\alpha_{3} + 64\alpha_{2} + 8\alpha_{1} + \alpha_{0} \quad (4) \\
15540 &= 6561\alpha_{4} + 729\alpha_{3} + 81\alpha_{2} + 9\alpha_{1} + \alpha_{0} \quad (5)
\end{align*}
\]

By solving that system of equation, obtained:

\[
P_{4}(m) = \frac{222}{24}m^{4} - \frac{1110}{24}m^{3} + \frac{7770}{24}m^{2} - \frac{5328}{24}m + 222
\]

\[
= \frac{222}{24}(m^{4} - 10m^{3} + 35m^{2} - 50m + 24)
\]

\[
= \frac{222}{24}(m-1)(m-2)(m-3)(m-4)
\]

\[
= 222 \times C_{4}^{(m-1)}.
\]

Table 3 shows the results for \( n=5 \) and \( n=6 \).

From Table 3, it can be seen that the number of graphs obtained make a pattern as: \( N(G_{n,m,t}) = k \times C_{t-1}^{(m-1)} \), where for \( n=5 \), \( k_{5}=222 \), \( k_{6}=205 \), \( k_{7}=110 \), \( k_{8}=45 \), \( k_{9}=10 \), and \( k_{10}=1 \). For \( n=6 \), \( k_{5}=1296 \), \( k_{6}=1980 \), \( k_{7}=3330 \), \( k_{8}=4620 \), \( k_{9}=6660 \), \( k_{10}=2640 \), \( k_{11}=1155 \), \( k_{12}=420 \), \( k_{13}=150 \), \( k_{14}=15 \), \( k_{15}=1 \). Note that for \( n=5 \), maximal number of \( t \) is 10, and maximal number of \( t \) for \( n=6 \) is 15. Thus, the formula for the number of connected vertices labeled graph of order five and six without loops differs only on the coefficient of \( t \) for every \( t \), and when \( t \) is maximum then the formula is \( C_{t-1}^{(m-1)} \).
Table 3. The Number of Connected Vertices Labeled Graphs of Order Five and Six without Parallel Edges.

<table>
<thead>
<tr>
<th>n</th>
<th>(N(G_{5,m})) = 222 (\times ) (C_{4}^{(m-1)})</th>
<th>(N(G_{6,m})) = 1296 (\times ) (C_{5}^{(m-1)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>(N(G_{5,5})) = 222 (\times ) (C_{4}^{(5-1)})</td>
<td>(N(G_{6,5})) = 1296 (\times ) (C_{5}^{(5-1)})</td>
</tr>
<tr>
<td>6</td>
<td>(N(G_{5,6})) = 205 (\times ) (C_{5}^{(6-1)})</td>
<td>(N(G_{6,6})) = 1980 (\times ) (C_{6}^{(6-1)})</td>
</tr>
<tr>
<td>7</td>
<td>(N(G_{5,7})) = 110 (\times ) (C_{6}^{(7-1)})</td>
<td>(N(G_{6,7})) = 3330 (\times ) (C_{6}^{(7-1)})</td>
</tr>
<tr>
<td>8</td>
<td>(N(G_{5,8})) = 45 (\times ) (C_{7}^{(8-1)})</td>
<td>(N(G_{6,8})) = 4620 (\times ) (C_{7}^{(8-1)})</td>
</tr>
<tr>
<td>9</td>
<td>(N(G_{5,9})) = 10 (\times ) (C_{8}^{(9-1)})</td>
<td>(N(G_{6,9})) = 6660 (\times ) (C_{8}^{(9-1)})</td>
</tr>
<tr>
<td>10</td>
<td>(N(G_{5,10})) = 1 (\times ) (C_{9}^{(10-1)})</td>
<td>(N(G_{6,10})) = 2640 (\times ) (C_{9}^{(10-1)})</td>
</tr>
<tr>
<td>11</td>
<td>(N(G_{6,11})) = 1155 (\times ) (C_{10}^{(11-1)})</td>
<td>(N(G_{6,11})) = 1155 (\times ) (C_{10}^{(11-1)})</td>
</tr>
<tr>
<td>12</td>
<td>(N(G_{6,12})) = 420 (\times ) (C_{11}^{(12-1)})</td>
<td>(N(G_{6,12})) = 150 (\times ) (C_{12}^{(12-1)})</td>
</tr>
<tr>
<td>13</td>
<td>(N(G_{6,13})) = 15 (\times ) (C_{13}^{(13-1)})</td>
<td>(N(G_{6,13})) = 2 (\times ) (C_{14}^{(13-1)})</td>
</tr>
<tr>
<td>14</td>
<td>(N(G_{6,14})) = 1 (\times ) (C_{14}^{(14-1)})</td>
<td>(N(G_{6,14})) = 1 (\times ) (C_{14}^{(14-1)})</td>
</tr>
</tbody>
</table>

CONCLUSIONS AND SUGGESTIONS

Based on the discussion above, it can be concluded that the formula for the number of connected vertices labeled graphs of order five and six is \(N(G_{n,m,t}) = k_{t} \cdot C_{t-1}^{(m-1)}\), where for \(n = 5\), \(k_{5} = 222\), \(k_{6} = 205\), \(k_{7} = 110\), \(k_{9} = 45\), \(k_{10} = 10\), and \(k_{11} = 1\). For \(n = 6\), \(k_{5} = 1296\), \(k_{6} = 1980\), \(k_{7} = 3330\), \(k_{8} = 4620\), \(k_{9} = 6660\), \(k_{10} = 2640\), \(k_{11} = 1155\), \(k_{12} = 420\), \(k_{13} = 150\), \(k_{14} = 15\), \(k_{15} = 1\). Moreover, it is concluded that if \(t\) is maximum, then for every order (five and six), \(N(G_{n,m,t}) = C_{t-1}^{(m-1)}\).

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