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Quasi-associative algebras on the frobenius lie algebra M_3 (R)⊕gl_3 (R)

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Abstract
In this paper, we study the quasi-associative algebra property for the real Frobenius Lie algebra $\mathfrak{g}_3 := \mathrm{M}_3(\mathbb{R}) \rtimes \mathfrak{gl}_3(\mathbb{R}) \subseteq \mathfrak{gl}_6(\mathbb{R})$ of dimension 18. The work aims to prove that $\mathfrak{g}_3$ is a quasi-associative algebra and to compute its formulas explicitly. To achieve this aim, we apply the literature reviews method corresponding to Frobenius Lie algebras, Frobenius functionals, and the structures of quasi-associative algebras. In the first step, we choose a Frobenius functional determined by direct computations of a bracket matrix of $\mathfrak{g}_3$ and in the second step, using an induced symplectic structure, we obtain the explicit formulas of quasi-associative algebras for $\mathfrak{g}_3$. As the results, we proved that $\mathfrak{g}_3$ has the quasi-associative algebras property, and we gave their formulas explicitly. For future research, the case of the quasi-associative algebras on $\mathrm{M}_{n,p}(\mathbb{R}) \rtimes \mathfrak{gl}_n(\mathbb{R})$ is still an open problem to be investigated. Our result can motivate to solve this problem.

INTRODUCTION

Quasi-associative algebras play an important role in many areas of physics and mathematics. They arise in the notions of vertex algebras, convex homogeneous cones, affine manifolds, and left-invariant affine structures (Burde, 2015). We can also describe how quasi-associative algebras arise in the representation theory of Lie groups and in the invariant theory of reductive groups. Therefore, it is an essential problem to consider whether a Lie algebra $\mathfrak{g}$ has a quasi-associative algebra structures and to give their explicit formulas of such structures. Roughly speaking, a quasi-associative algebra structure or a left-symmetric algebra structure on a vector space $\mathfrak{g}$ is a bilinear product $*: \mathfrak{g} \times \mathfrak{g} \ni (\alpha, \beta) \mapsto \alpha * \beta \in \mathfrak{g}$ such that for each $x, y, z \in \mathfrak{g}$, the associator $\varphi(x, y, z)$ is left-symmetric (Diatta, A., Manga, & Mbaye, 2020). Moreover, if the vector space $\mathfrak{g}$ is a Lie algebra, then the quasi-associative algebra $*$ is compatible with its Lie brackets of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ which has these both structures is called a quasi-associative algebra. We recall that not all Lie algebras or Lie groups are quasi-associative algebras. For example, the n-dimensional filiform nilpotent Lie groups where $10 \leq n \leq 13$ do not admit quasi-associative algebra structures (Burde, 2015). Furthermore, we can also see that any semi-simple Lie algebra has no quasi-associative algebra structures. On the other hand, the Lie algebra of the 4-dimensional Lie similitude group has quasi-associative algebra structures (Kurniadi, Gusriani, & Subartini, 2020) and so has affine Lie algebras of dimension 6 (Hendrawan, 2020).

In the theory of Lie algebras, we can find some types of Lie algebras. It is well known the notion of 6-dimensional nilpoten Lie algebras (Graaf, 2007) and filiform Lie algebras (Hadjer & Makhlouf, 2012). Another important class of Lie algebras is a Frobenius Lie algebras (Alvarez & et al, 2018). Many researchers investigated this Frobenius Lie algebra. For example, we can study the principal elements of Frobenius Lie algebras (Diatta & Manga,

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2014), $\mathfrak{g}$ −quasi Frobenius Lie algebras (Pham, 2016), and the properties of Frobenius Lie algebras with abelian nilradical (Alvarez & et al, 2018).

In the present work, we aim to prove quasi-associative algebra structures for the Lie algebra $\mathfrak{g}_3 := \mathrm{M}_3(\mathbb{R}) \rtimes \mathfrak{gl}_3(\mathbb{R})$. This Lie algebra is the special case for $n = p = 3$ of the Lie algebra $\mathfrak{g} := \mathrm{M}_{n,p}(\mathbb{R}) \rtimes \mathfrak{gl}(n, \mathbb{R})$ where $\mathrm{M}_{n,p}(\mathbb{R})$ is the vector space of $n \times p$ matrices and $\mathfrak{gl}(n, \mathbb{R})$ is the Lie algebra of $n \times n$ matrices (Rais, 1978). It has been proven that $\mathfrak{g} := \mathrm{M}_{n,p}(\mathbb{R}) \rtimes \mathfrak{gl}(n, \mathbb{R})$ is the Frobenius Lie algebra if p divides n (Rais, 1978). We believe that $\mathfrak{g}_3$ is the ideal situation where the quasi-associative algebra structures can be computed perfectly. It gives us to have simple and visual computations to all the existence of quasi-associative algebra structures for Frobenius Lie algebras. Furthermore, we can generalize our result to case the Frobenius Lie algebra $\mathfrak{g} := \mathrm{M}_{n,p}(\mathbb{R}) \rtimes \mathfrak{gl}(n, \mathbb{R})$.

We summarize our result as follows. Let $\mathfrak{g}_3 := \mathrm{M}_3(\mathbb{R}) \rtimes \mathfrak{gl}_3(\mathbb{R})$ be the Frobenius Lie algebra of dimension 18. Then $\mathfrak{g}_3$ admits quasi-associative algebra structures and we compute their structures explicitly. Since the Lie algebra $\mathfrak{g}_3$ is Frobenius, we can choose a Frobenius functional $\psi_0 \in \mathfrak{g}_3^*$ which corresponds to an alternating bilinear form B_{ψ_0}. Indeed, we can observe that B_{ψ_0} is non-degenerate. Moreover, we apply symplectic form to induce the quasi-associative algebra structure on $\mathfrak{g}_3$ (Diatta, Manga, & Mbaye, 2020). Thus, we obtain the explicit formulas for quasi-associative algebra structures on $\mathfrak{g}_3$.
We organize the paper as follows. In introduction section, we devote the background, motivation, the purpose, and statement of main result of research. In research methods section, we applied some literature reviews method for this research. Finally, a complete proof of the existence of quasi-associative algebra structures for $\mathfrak{g}_3$ is given in result and discussion section. The paper ends by conclusion and suggestion.

METHODS

The method used in this research is literature review, especially about Frobenius Lie algebra $\mathfrak{g}_3$ of dimension 18 and about the quasi-associative algebra structures on the Lie algebra $\mathfrak{g}_3$ (Burde, 2015). First of all, given Lie algebra $\mathfrak{g}_3$, it can be proven that the existence of quasi-associative algebra structure on the Frobenius Lie algebra $\mathfrak{g}_3$ and it is given quasi-associative algebras structures formula explicitly.

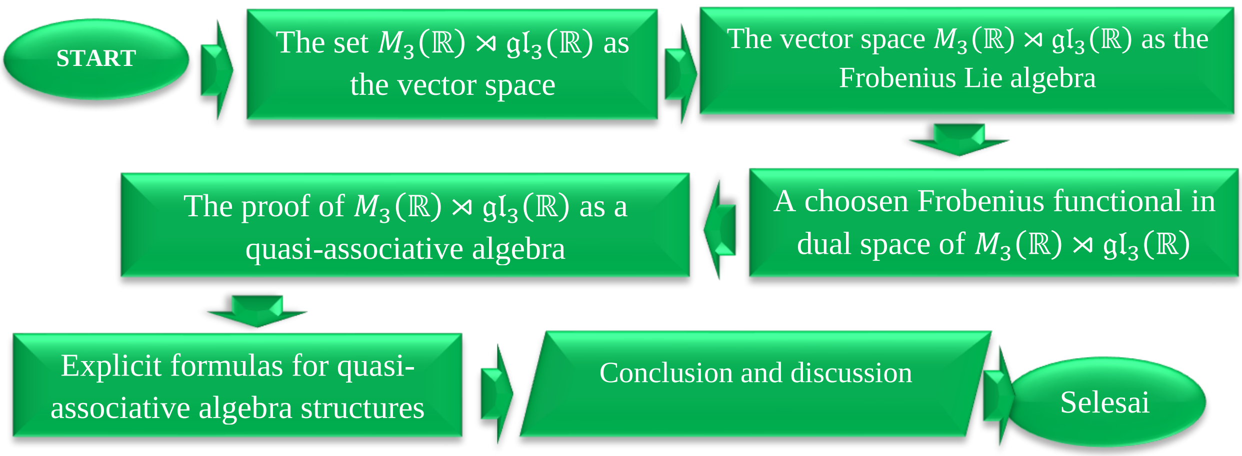

Chart 1. Research flow

RESULTS AND DISCUSSION

Before discussing our main result, we will briefly review some fundamental principles that will be helpful in our article. We review some notions of Lie algebras, Frobenius functional, and quasi-associative algebras. Let $\mathbb{R}$ be the set of all real numbers. The Lie algebra $\mathfrak{g}_3 := \mathrm{M}_3(\mathbb{R}) \rtimes \mathfrak{gl}_3(\mathbb{R})$ stated in this paper is the semi-direct sum of the vector space $\mathrm{M}_3(\mathbb{R})$ and the Lie algebra $\mathfrak{gl}_3(\mathbb{R})$ consisting of real matrices 3×3. In the specific realization, the Lie algebra $\mathfrak{g}_3$ can be written in the following matrix form.

$$\mathfrak{g}_3 = \left\{ \begin{pmatrix} a_1 & b_1 & c_1 & u_1 & v_1 & w_1 \\ a_2 & b_2 & c_2 & u_2 & v_2 & w_2 \\ a_3 & b_3 & c_3 & u_3 & v_3 & w_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \; ; \; a_i, b_i, c_i, u_i, v_i, w_i \in \mathbb{R}, i = 1,2,3 \right\} \subseteq \mathfrak{gl}_6(\mathbb{R}). \quad (1)$$

In another realization, the equation (1) can be written in the following block matrix form

$$\mathfrak{g}_3 = \left\{ \begin{pmatrix} \alpha & U \\ O & O \end{pmatrix} \; ; \; \alpha \in \mathfrak{gl}_3(\mathbb{R}), U \in \mathrm{M}_3(\mathbb{R}) \right\} \subseteq \mathfrak{gl}_6(\mathbb{R}). \quad (2)$$

where the matrices α and U are of the forms

$\alpha := \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \in \mathfrak{gl}_3(\mathbb{R})$ and $U := \begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix} \in \mathrm{M}_3(\mathbb{R})$. Furthermore, using the standard basis for $\mathfrak{g}_3$, we have that $\mathfrak{g}_3 = \mathrm{Span}\, S$ where

$$S = \left\{ x_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \dots, x_9 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, x_{10} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \dots, x_{18} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}. \quad (3)$$

The Lie brackets for the Lie algebra $\mathfrak{g}_3$ is determined by the commutator matrix as follows

$$[x_i, x_j] = x_i x_j - x_j x_i, \quad x_i, x_j \in \mathfrak{g}_3, i, j = 1,2,3, \dots, 18. \quad (4)$$

The detail computations of these Lie brackets for $\mathfrak{g}_3$ have been done in (Henti et al, 2021).

Let x, y, z be elements of a Lie algebra. The associators of three elements x, y, z are defined by

$$\varphi(x, y, z) = (xy)z - x(yz) \quad (5)$$

The quasi-associative algebra occurs when it is filled

$$\varphi(x, y, z) = \varphi(y, x, z)$$

$$[x, y] = xy - yx \tag{6}$$

Definition 1 (Hilgert & Neeb, 2012). *Let* $\mathfrak{g}$ *be a real vector space and* $[.,.]: \mathfrak{g} \times \mathfrak{g} \ni (a, b) \mapsto [a, b] \in \mathfrak{g}$ *be a bilinear form. The bilinear form* $[.,.]$ *is called a Lie bracket for* $\mathfrak{g}$ *if the following conditions are satisfied*:

1. $[x, y] = -[y, x]; \;\forall\, x, y \in \mathfrak{g}$,
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$; for $x, y, z \in \mathfrak{g}$.

The last condition is called the Jacobi identity. Furthermore, the real vector space $\mathfrak{g}$ *equipped by Lie brackets is called a Lie algebra.*

Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{g}^*$ be its dual vector space of $\mathfrak{g}$ consisting of all linear functionals σ in the Lie algebra $\mathfrak{g}$. In other words, $\mathfrak{g}^*$ can be written in the following form

$$\mathfrak{g}^* \coloneqq \{\mu_0 \mid \mu_0 : \mathfrak{g} \to \mathbb{R} \text{ is a linear functional}\} \tag{7}$$

Definition 2 (Ooms, 1980) *A real Lie algebra* $\mathfrak{g}$ *is said to be Frobenius if the stabilzer of* $\mathfrak{g}$ *defined by*

$$\mathfrak{g}^{\psi_0} = \{x \in \mathfrak{g}\,; \langle \psi_0, [x, y] \rangle = 0\,, \forall\, y \in \mathfrak{g}\}. \tag{8}$$

is trivial for some linear functionals $\psi_0 \in \mathfrak{g}^*$.

The linear functional ψ_0 satisfying the equation (8) is called the Frobenius functional. The example of Frobenius Lie algebra can be seen in the Theorem below which is also the object of this research.

Proposition 3 (Henti et al., 2021) *Let* $\mathfrak{g}_3 \coloneqq \mathrm{M}_3(\mathbb{R}) \rtimes \mathfrak{gl}_3(\mathbb{R})$ *be a Lie algebra of dimension 18 whose basis is* $S \coloneqq \{x_i\}_{i=1}^{18}$. *Then there exists the Frobenius functional* $\psi_0 = x_{10}^* + x_{14}^* + x_{18}^*$ *such that* $\mathfrak{g}^{\psi_0} = \{0\}$. *Therefore,* $\mathfrak{g}_3$ *is the Frobenius Lie algebra.*

Other examples of Frobenius Lie algebras can be seen in the work of (Csikós & Verhóczki, 2007). He classified 4-dimensional Frobenius Lie algebras over a filed of characteristic $\neq 2$ and 6-dimensional Frobenius Lie algebras over an algebraic closed field.

Let $\mathfrak{g}$ Lie algebra with basis $S = \{x_i\}_{i=1}^{2n}$. We denoted by the $2n \times 2n$ matrix $\mathfrak{N}([x_i, x_j])$ whose (i, j)-th entry is determined by the Lie bracket of $\mathfrak{g}$ where $1 \le i, j \le 2n$ and $\mathfrak{N}(\psi_0)$ be a matrix whose (i, j)-th entry is determined by $\langle \psi_0, [x_i, x_j] \rangle_{1 \le i, j \le 2n}$.

Theorem 4 (Ooms, 1980) *Let* $\mathfrak{g}$ *be a Lie algebra with basis* $S = \{x_i\}_{i=1}^{2n}$. *If there exist a linear functional* ψ_0 *such that the determinant of* $\mathfrak{N}(\psi_0)$ *isn't equal to 0 then* $\mathfrak{g}$ *is called a Frobenius Lie algebra.*

Furthermore, the basic notations needed in this study is quasi-associative algebras which can be explained as follows

Theorem 5 (Burde, 2015) *Let* $\mathfrak{g}$ *be a Lie algebra. The quasi-associative algebra structure on* $\mathfrak{g}$ *is bilinear product*

$$\mathfrak{g} \times \mathfrak{g} \ni (a, b) \mapsto a * b \in \mathfrak{g}, \tag{9}$$

which satisfies the equation (6).

Especially for Frobenius Lie algebra, the existence of the Frobenius functional $\psi_0 \in \mathfrak{g}^*$ can be used to find real bilinear products that satisfies the equation (6). On the other hand, in

Frobenius Lie algebra the determinant of matrix $\mathfrak{N}(\psi_0)$ is the same as the determinant of the alternating bilinear form,

$$B_{\psi_0}:\ \mathfrak{g}\times\mathfrak{g}\ni(x_i,x_j)\mapsto\langle\psi_0,[x_i,x_j]\rangle\in\mathbb{R} \tag{10}$$

with respect to the basis $\{x_i\}$.

Furthermore, it has been proven in (Diatta, A. et al., 2020) that a linear symplectic structure form ϕ on Lie algebra $\mathfrak{g}$ can induce the equation (6). Let B_{ψ_0} be an alternating bilinear form on a Frobenius Lie algebra $\mathfrak{g}$ where ψ_0 is the Frobenius functional. Furthermore, B_{ψ_0} is a symplectic form which induces a quasi-associative algebra (Diatta, A. et al., 2020). It is defined by

$$B_{\psi_0}(x*y,z):=-B_{\psi_0}(y,[x,z])=-\langle\psi_0,[y,[x,z]]\rangle \tag{11}$$

for all $x,y,z\in\mathfrak{g}$.

In this section, we shall prove our main results that $\mathfrak{g}_3$ admits quasi-associative algebra structures. Our main result is stated in the following Proposition

Proposition 7. *The Frobenius Lie algebra $\mathfrak{g}_3$ with the standard basis $S=\{x_i\}_{i=1}^{18}$ has quasi-associative algebra structures.*

Proof. Firstly, we define an alternating bilinear form of $\mathfrak{g}_3$

$$B_{\psi_0}:\mathfrak{g}_3\times\mathfrak{g}_3\ni(x_i,x_j)\mapsto\langle\psi_0,[x_i,x_j]\rangle\in\mathbb{R}\ ,1\leq i,j\leq 18. \tag{12}$$

with respect to basis $\{x_i\}_{i=1}^{18}$. Since $\mathfrak{g}_3$ is Frobenius Lie algebra, then there exists a Frobenius functional $\psi_0\in\mathfrak{g}_3^*$ such that B_{ψ_0} is symplectic form. In this case, we choose the Frobenius functional $\psi_0\coloneqq x_{10}^*+x_{14}^*+x_{18}^*\in\mathfrak{g}_3^*$. Furthermore, let u,v be elements in $\mathfrak{g}_3$ and we define a bilinear product $\mathfrak{g}_3\times\mathfrak{g}_3\ni(u,v)\mapsto u*v\in\mathfrak{g}_3$. Since $\mathfrak{g}_3=\operatorname{Span}S$, then for each u and v in $\mathfrak{g}_3$, there exist scalars α_i,β_j, and $\gamma_k\in\mathbb{R}$ where $1\leq i,j,k\leq 18$ such that

$$u\coloneqq\sum_{i=1}^{18}\alpha_i x_i\ ,v\coloneqq\sum_{j=1}^{18}\beta_j x_j\ ,u*v\coloneqq\sum_{k=1}^{18}\gamma_k x_k. \tag{13}$$

Using the equations (10) and (11), we have

$$B_{\psi_0}(u*v,x_i)=-B_{\psi_0}(v,[u,x_i]),\ 1\leq i\leq 18. \tag{14}$$

where $\psi_0(x_i)=1$ if $i=10,14,18$ and zero for the others. Since the equation (14) is a symplectic form then it induces a quasi-associative algebra structures on $\mathfrak{g}_3$ in the equation (6). To prove $B_{\psi_0}(u*v,x_i)$ in the equation (14) induces $\varphi(u,v,x_i)=\varphi(v,u,x_i)$ for each $i=1,2,3,\dots,18$, we shall prove that $(u*v)*x_i-(v*u)*x_i=u*(v*x_i)-v*(u*x_i)$. Using the equation (14), then for each $k\in\mathfrak{g}_3$, we have

$$\begin{aligned}
&B_{\psi_0}\big((u*v)*x_i-(v*u)*x_i,k\big)=B_{\psi_0}\big((u*v)*x_i,k\big)-B_{\psi_0}((v*u)*x_i,k)\\
&=B_{\psi_0}(x_i,[v*u,k])-B_{\psi_0}(x_i,[u*v,k])\\
&=B_{\psi_0}(x_i,[v*u,k]-[u*v,k])\\
&=B_{\psi_0}(x_i,[v*u-u*v,k])\\
&=B_{\psi_0}(x_i,[[v,u],k])
\end{aligned}$$

In the similar way, using the identity Jacobi, we also have $B_{\psi_0}\big(u*(v*x_i)-v*(u*x_i)\big)=B_{\psi_0}(x_i,[[v,u],k])$. Furthermore, since B_{ψ_0} is nondegenerate, then we have $\varphi(u,v,x_i)=$

$\varphi(v, u, x_i)$. Now, we shall prove that the equation (14) induces $[u, v] = u * v - v * u$. Following the computations before, we also obtain $B_{\psi_0}([u, v] - u * v + v * u, k) = 0$ for each $k \in \mathfrak{g}_3$. since B_{ψ_0} is nondegenerate, then we get $[u, v] = u * v - v * u$.

In the other words we can find the solutions γ_i in the terms of α_i and β_i where $1 \leq i \leq 18$. Namely, we have

$$u * v = \left(\sum_{i=1}^{18} \alpha_i x_i\right) * \left(\sum_{j=1}^{18} \beta_j x_j\right) = \sum_{k=1}^{18} \gamma_k x_k \qquad (15)$$

To construct the product $x_i * x_j$ in $\mathfrak{g}_3$ where $1 \leq i, j < 18$, the corresponding values of α_i and β_j are assigned such that the equation (14) satisfies the equation (6). Therefore, the equations (14) and (15) induces the equation (6). Thus, $\mathfrak{g}_3$ has quasi-associative algebra structures as required.

■

Now, we give the explicit formulas for the quasi-associative algebra structures on $\mathfrak{g}_3$. We apply the equations (13, (14), and (15) to compute γ_k in the terms α_i and β_i $1 \leq i, j, k \leq 18$, then we obatin the following formulas

$$\begin{aligned}
\gamma_{10} &= \alpha_2\beta_{13} + \alpha_3\beta_{16} - \alpha_4\beta_{11} - \alpha_7\beta_{12} - \alpha_{10}\beta_1 - \alpha_{11}\beta_4 - \alpha_{12}\beta_7 \\
\gamma_{13} &= -\alpha_1\beta_{13} - \alpha_4\beta_{14} + \alpha_4\beta_{10} + \alpha_5\beta_{13} + \alpha_6\beta_{16} - \alpha_7\beta_{15} - \alpha_{13}\beta_1 - \alpha_{14}\beta_4 - \alpha_{15}\beta_7 \\
\gamma_{16} &= -\alpha_{16}\beta_1 - \alpha_{17}\beta_4 - \alpha_{18}\beta_7 + \alpha_7\beta_{10} + \alpha_8\beta_{13} - \alpha_1\beta_{16} + \alpha_9\beta_{16} - \alpha_4\beta_{17} - \alpha_7\beta_8 \\
\gamma_{11} &= \alpha_1\beta_{11} - \alpha_2\beta_{10} + \alpha_2\beta_{14} + \alpha_3\beta_{17} - \alpha_5\beta_{11} - \alpha_8\beta_{12} - \alpha_{10}\beta_2 - \alpha_{11}\beta_5 - \alpha_{12}\beta_8 \\
\gamma_{14} &= -\alpha_2\beta_{13} + \alpha_4\beta_{11} + \alpha_6\beta_{17} - \alpha_8\beta_{15} - \alpha_{13}\beta_2 - \alpha_{14}\beta_5 - \alpha_{15}\beta_8 \\
\gamma_{17} &= -\alpha_2\beta_{16} - \alpha_5\beta_{17} + \alpha_7\beta_{11} - \alpha_8\beta_{18} + \alpha_8\beta_{14} + \alpha_9\beta_{17} - \alpha_{16}\beta_2 - \alpha_{17}\beta_5 - \alpha_{18}\beta_8 \\
\gamma_{12} &= \alpha_1\beta_{12} + \alpha_2\beta_{15} - \alpha_3\beta_{10} + \alpha_3\beta_{18} - \alpha_6\beta_{11} - \alpha_9\beta_{12} - \alpha_{10}\beta_3 - \alpha_{11}\beta_6 - \alpha_{12}\beta_9 \\
\gamma_{15} &= -\alpha_3\beta_{13} + \alpha_4\beta_{12} + \alpha_5\beta_{15} - \alpha_6\beta_{14} + \alpha_6\beta_{18} - \alpha_9\beta_{15} - \alpha_{13}\beta_3 - \alpha_{14}\beta_6 - \alpha_{15}\beta_9 \\
\gamma_{18} &= -\alpha_3\beta_{16} - \alpha_6\beta_{17} + \alpha_7\beta_{12} + \alpha_8\beta_{15} - \alpha_{16}\beta_3 - \alpha_{17}\beta_6 - \alpha_{18}\beta_9 \\
\gamma_1 &= -\alpha_1\beta_1 - \alpha_4\beta_2 - \alpha_7\beta_3 \\
\gamma_4 &= -\alpha_1\beta_4 - \alpha_4\beta_5 - \alpha_7\beta_6 \\
\gamma_7 &= -\alpha_1\beta_7 - \alpha_4\beta_8 - \alpha_7\beta_9 \\
\gamma_2 &= -\alpha_2\beta_1 - \alpha_5\beta_2 - \alpha_8\beta_3 \\
\gamma_5 &= -\alpha_2\beta_4 - \alpha_5\beta_5 - \alpha_8\beta_6 \\
\gamma_8 &= -\alpha_2\beta_7 - \alpha_5\beta_8 - \alpha_8\beta_9 \\
\gamma_3 &= -\alpha_3\beta_1 - \alpha_6\beta_2 - \alpha_9\beta_3 \\
\gamma_6 &= -\alpha_3\beta_4 - \alpha_6\beta_5 - \alpha_9\beta_6 \\
\gamma_9 &= -\alpha_3\beta_7 - \alpha_6\beta_8 - \alpha_9\beta_9
\end{aligned} \qquad (16)$$

Therefore, in the terms α_i and β_j, we can rewrite the equation (15) as follows.

$$
\begin{aligned}
u * v = \left(\sum_{i=1}^{18} \alpha_i x_i\right) * \left(\sum_{j=1}^{18} \beta_j x_j\right) = \sum_{k=1}^{18} \gamma_k x_k \\
&= (-\alpha_1\beta_1 - \alpha_4\beta_2 - \alpha_7\beta_3)x_1 + (-\alpha_2\beta_1 - \alpha_5\beta_2 - \alpha_8\beta_3)x_2 \\
&+ (-\alpha_3\beta_1 - \alpha_6\beta_2 - \alpha_9\beta_3)x_3 + (-\alpha_1\beta_4 - \alpha_4\beta_5 - \alpha_7\beta_6)x_4 \\
&+ (-\alpha_2\beta_4 - \alpha_5\beta_5 - \alpha_8\beta_6)x_5 + (-\alpha_3\beta_4 - \alpha_6\beta_5 - \alpha_9\beta_6)x_6 \\
&+ (-\alpha_1\beta_7 - \alpha_4\beta_8 - \alpha_7\beta_9)x_7 + (-\alpha_2\beta_7 - \alpha_5\beta_8 - \alpha_8\beta_9)x_8 \\
&+ (-\alpha_3\beta_7 - \alpha_6\beta_8 - \alpha_9\beta_9)x_9 \\
&+ (\alpha_2\beta_{13} + \alpha_3\beta_{16} - \alpha_4\beta_{11} - \alpha_7\beta_{12} - \alpha_{10}\beta_1 - \alpha_{11}\beta_4 - \alpha_{12}\beta_7)x_{10} \\
&+ (\alpha_1\beta_{11} - \alpha_2\beta_{10} + \alpha_2\beta_{14} + \alpha_3\beta_{17} - \alpha_5\beta_{11} - \alpha_8\beta_{12} - \alpha_{10}\beta_2 - \alpha_{11}\beta_5 \\
&- \alpha_{12}\beta_8)x_{11} \\
&+ (\alpha_1\beta_{12} + \alpha_2\beta_{15} - \alpha_3\beta_{10} + \alpha_3\beta_{18} - \alpha_6\beta_{11} - \alpha_9\beta_{12} - \alpha_{10}\beta_3 - \alpha_{11}\beta_6 \\
&- \alpha_{12}\beta_9)x_{12} \\
&+ (-\alpha_1\beta_{13} - \alpha_4\beta_{14} + \alpha_4\beta_{10} + \alpha_5\beta_{13} + \alpha_6\beta_{16} - \alpha_7\beta_{15} - \alpha_{13}\beta_1 - \alpha_{14}\beta_4 \\
&- \alpha_{15}\beta_7)x_{13} \\
&+ (-\alpha_2\beta_{13} + \alpha_4\beta_{11} + \alpha_6\beta_{17} - \alpha_8\beta_{15} - \alpha_{13}\beta_2 - \alpha_{14}\beta_5 - \alpha_{15}\beta_8)x_{14} \\
&+ (-\alpha_3\beta_{13} + \alpha_4\beta_{12} + \alpha_5\beta_{15} - \alpha_6\beta_{14} + \alpha_6\beta_{18} - \alpha_9\beta_{15} - \alpha_{13}\beta_3 - \alpha_{14}\beta_6 \\
&- \alpha_{15}\beta_9)x_{15} \\
&+ (-\alpha_{16}\beta_1 - \alpha_{17}\beta_4 - \alpha_{18}\beta_7 + \alpha_7\beta_{10} + \alpha_8\beta_{13} - \alpha_1\beta_{16} + \alpha_9\beta_{16} - \alpha_4\beta_{17} \\
&- \alpha_7\beta_8)x_{16} \\
&+ (-\alpha_2\beta_{16} - \alpha_5\beta_{17} + \alpha_7\beta_{11} - \alpha_8\beta_{18} + \alpha_8\beta_{14} + \alpha_9\beta_{17} - \alpha_{16}\beta_2 - \alpha_{17}\beta_5 \\
&- \alpha_{18}\beta_8)x_{17} \\
&+ (-\alpha_3\beta_{16} - \alpha_6\beta_{17} + \alpha_7\beta_{12} + \alpha_8\beta_{15} - \alpha_{16}\beta_3 - \alpha_{17}\beta_6 - \alpha_{18}\beta_9)x_{18}
\end{aligned} \tag{17}
$$

By considering the corresponding values of α_i and β_j where $1 \le i, j \le 18$, then we obtain the explicit formulas of quasi-associative algebra structures as follows

Table 1. The Quasi-Associative Algebra Structures for $\mathfrak{g}_3$

The bilinear product $x_i * x_j$, $1 \le i, j \le 18$		
$x_1 * x_1 = -x_1$	$x_5 * x_5 = -x_5$	$x_9 * x_6 = -x_6$
$x_1 * x_4 = -x_4$	$x_5 * x_8 = -x_8$	$x_9 * x_9 = -x_9$
$x_1 * x_7 = -x_7$	$x_5 * x_{11} = -x_{11}$	$x_9 * x_{12} = -x_{12}$
$x_1 * x_{11} = x_{11}$	$x_5 * x_{13} = x_{13}$	$x_9 * x_{15} = -x_{15}$
$x_1 * x_{12} = x_{12}$	$x_5 * x_{15} = x_{15}$	$x_9 * x_{16} = x_{16}$
$x_1 * x_{13} = -x_{13}$	$x_5 * x_{17} = -x_{17}$	$x_9 * x_{17} = x_{17}$
The bilinear product $x_i * x_j$, $1 \le i, j \le 18$		
$x_1 * x_{16} = -x_{16}$	$x_6 * x_2 = -x_3$	$x_{10} * x_1 = -x_{10}$
$x_2 * x_1 = -x_2$	$x_6 * x_5 = -x_6$	$x_{10} * x_2 = -x_{11}$
$x_2 * x_4 = -x_5$	$x_6 * x_8 = -x_9$	$x_{10} * x_3 = -x_{12}$
$x_2 * x_7 = -x_8$	$x_6 * x_{11} = -x_{12}$	$x_{11} * x_4 = -x_{10}$
$x_2 * x_{10} = -x_{11}$	$x_6 * x_{14} = -x_{15}$	$x_{11} * x_5 = -x_{11}$
$x_2 * x_{13} = x_{10} - x_{14}$	$x_6 * x_{16} = x_{13}$	$x_{11} * x_6 = -x_{12}$
$x_2 * x_{14} = x_{11}$	$x_6 * x_{17} = x_{14} - x_{18}$	$x_{12} * x_7 = -x_{10}$
$x_2 * x_{15} = x_{12}$	$x_6 * x_{18} = x_{15}$	$x_{12} * x_8 = -x_{11}$

$x_2 * x_{16} = -x_{17}$
$x_3 * x_1 = -x_3$
$x_3 * x_4 = -x_6$
$x_3 * x_7 = -x_9$
$x_3 * x_{10} = -x_{12}$
$x_3 * x_{13} = -x_{15}$
$x_3 * x_{16} = x_{10} - x_{18}$
$x_3 * x_{17} = x_{11}$
$x_3 * x_{18} = x_{12}$
$x_4 * x_2 = -x_1$
$x_4 * x_5 = -x_4$
$x_4 * x_8 = -x_7$
$x_4 * x_{10} = x_{13}$
$x_4 * x_{11} = x_{14} - x_{10}$
$x_4 * x_{12} = x_{15}$
$x_4 * x_{14} = -x_{13}$
$x_4 * x_{17} = -x_{16}$
$x_5 * x_2 = -x_2$
$x_7 * x_3 = -x_1$
$x_7 * x_6 = -x_4$
$x_7 * x_9 = -x_7$
$x_7 * x_{10} = x_{16}$
$x_7 * x_{11} = x_{17}$
$x_7 * x_{12} = x_{18} - x_{10}$
$x_7 * x_{15} = -x_{13}$
$x_7 * x_{18} = -x_{16}$
$x_8 * x_3 = -x_2$
$x_8 * x_6 = -x_5$
$x_8 * x_9 = -x_8$
$x_8 * x_{12} = -x_{11}$
$x_8 * x_{13} = x_{16}$
$x_8 * x_{14} = x_{17}$
$x_8 * x_{15} = x_{18} - x_{14}$
$x_8 * x_{18} = -x_{17}$
$x_9 * x_3 = -x_3$
$x_{18} * x_9 = -x_{18}$
$x_{12} * x_9 = -x_{12}$
$x_{13} * x_1 = -x_{13}$
$x_{13} * x_2 = -x_{14}$
$x_{13} * x_3 = -x_{15}$
$x_{14} * x_4 = -x_{13}$
$x_{14} * x_5 = -x_{14}$
$x_{14} * x_6 = -x_{15}$
$x_{15} * x_7 = -x_{13}$
$x_{15} * x_8 = -x_{14}$
$x_{15} * x_9 = -x_{15}$
$x_{16} * x_1 = -x_{16}$
$x_{16} * x_2 = -x_{17}$
$x_{16} * x_3 = -x_{18}$
$x_{17} * x_4 = -x_{16}$
$x_{17} * x_5 = -x_{17}$
$x_{17} * x_6 = -x_{18}$
$x_{18} * x_7 = -x_{16}$
$x_{18} * x_8 = -x_{17}$

$$
\begin{aligned}
x_1 * x_2 &= x_1 * x_3 = x_1 * x_5 = x_1 * x_6 = x_1 * x_8 = x_1 * x_9 = x_1 * x_{10} = x_1 * x_{14} \\
&= x_1 * x_{15} = x_1 * x_{17} = x_1 * x_{18} = x_2 * x_2 = x_2 * x_3 = x_2 * x_5 \\
&= x_2 * x_6 = x_2 * x_8 = x_2 * x_9 = x_2 * x_{11} = x_2 * x_{12} = x_2 * x_{17} \\
&= x_2 * x_{18} = x_3 * x_2 = x_3 * x_3 = x_3 * x_5 = x_3 * x_6 = x_3 * x_8 \\
&= x_3 * x_9 = x_3 * x_{11} = x_3 * x_{12} = x_3 * x_{14} = x_3 * x_{15} = x_4 * x_1 \\
&= x_4 * x_3 = x_4 * x_4 = x_4 * x_6 = x_4 * x_7 = x_4 * x_9 = x_4 * x_{13} \\
&= x_4 * x_{15} = x_4 * x_{16} = x_4 * x_{18} = x_5 * x_1 = x_5 * x_3 = x_5 * x_4 \\
&= x_5 * x_6 = x_5 * x_7 = x_5 * x_9 = x_5 * x_{10} = x_5 * x_{12} = x_5 * x_{14} \\
&= x_5 * x_{16} = x_5 * x_{18} = x_6 * x_1 = x_6 * x_3 = x_6 * x_4 = x_6 * x_6 \\
&= x_6 * x_7 = x_6 * x_9 = x_6 * x_{10} = x_6 * x_{12} = x_6 * x_{13} = x_6 * x_{15} \\
&= x_7 * x_1 = x_7 * x_2 = x_7 * x_4 = x_7 * x_5 = x_7 * x_7 = x_7 * x_8 \\
&= x_7 * x_{13} = x_7 * x_{14} = x_7 * x_{16} = x_7 * x_{17} = x_8 * x_1 = x_8 * x_2 \\
&= x_8 * x_4 = x_8 * x_5 = x_8 * x_7 = x_8 * x_8 = x_8 * x_{10} = x_8 * x_{11} \\
&= x_8 * x_{16} = x_8 * x_{17} = x_9 * x_1 = x_9 * x_2 = x_9 * x_4 = x_9 * x_5 \\
&= x_9 * x_7 = x_9 * x_8 = x_9 * x_{10} = x_9 * x_{11} = x_9 * x_{13} = x_9 * x_{14} \\
&= x_9 * x_{18} = x_{10} * x_4 = x_{10} * x_5 = x_{10} * x_6 = x_{10} * x_7 = x_{10} * x_8 \\
&= x_{10} * x_9 = x_{10} * x_{10} = x_{10} * x_{11} = x_{10} * x_{12} = x_{10} * x_{13} \\
&= x_{10} * x_{14} = x_{10} * x_{15} = x_{10} * x_{16} = x_{10} * x_{17} = x_{10} * x_{18} \\
&= x_{11} * x_1 = x_{11} * x_2 = x_{11} * x_3 = x_{11} * x_7 = x_{11} * x_8 = x_{11} * x_9 \\
&= x_{11} * x_{10} = x_{11} * x_{11} = x_{11} * x_{12} = x_{11} * x_{13} = x_{11} * x_{14} \\
&= x_{11} * x_{15} = x_{11} * x_{16} = x_{11} * x_{17} = x_{11} * x_{18} = x_{12} * x_1 \\
&= x_{12} * x_2 = x_{12} * x_3 = x_{12} * x_4 = x_{12} * x_{15} = x_{12} * x_{16} \\
&= x_{12} * x_{17} = x_{12} * x_{18} = x_{13} * x_4 = x_{13} * x_5 = x_{13} * x_6 = x_{13} * x_7 \\
&= x_{13} * x_8 = x_{13} * x_9 = x_{13} * x_{10} = x_{13} * x_{11} = x_{13} * x_{12} \\
&= x_{13} * x_{13} = x_{13} * x_{14} = x_{13} * x_{15} = x_{13} * x_{16} = x_{13} * x_{17} \\
&= x_{13} * x_{18} = x_{14} * x_1 = x_{14} * x_2 = x_{14} * x_3 = x_{14} * x_7 = x_{14} * x_8 \\
&= x_{14} * x_9 = x_{14} * x_{10} = x_{14} * x_{11} = x_{14} * x_{12} = x_{14} * x_{13} \\
&= x_{14} * x_{14} = x_{14} * x_{15} = x_{14} * x_{16} = x_{14} * x_{17} = x_{14} * x_{18} \\
&= x_{15} * x_1 = x_{15} * x_2 = x_{15} * x_3 = x_{15} * x_4 = x_{15} * x_5 = x_{15} * x_6 \\
&= x_{15} * x_{10} = x_{15} * x_{11} = x_{15} * x_{12} = x_{15} * x_{13} = x_{15} * x_{14} \\
&= x_{15} * x_{15} = x_{15} * x_{16} = x_{15} * x_{17} = x_{15} * x_{18} = x_{16} * x_4 \\
&= x_{16} * x_5 = x_{16} * x_6 = x_{16} * x_7 = x_{16} * x_8 = x_{16} * x_9 = x_{16} * x_{10} \\
&= x_{16} * x_{11} = x_{16} * x_{12} = x_{16} * x_{13} = x_{16} * x_{14} = x_{16} * x_{15} \\
&= x_{16} * x_{16} = x_{16} * x_{17} = x_{16} * x_{18} = x_{17} * x_1 = x_{17} * x_2 \\
&= x_{17} * x_3 = x_{17} * x_7 = x_{17} * x_8 = x_{17} * x_9 = x_{17} * x_{10} = x_{17} * x_{11} \\
&= x_{17} * x_{12} = x_{17} * x_{13} = x_{17} * x_{14} = x_{17} * x_{15} = x_{17} * x_{16} \\
&= x_{17} * x_{17} = x_{17} * x_{18} = x_{18} * x_1 = x_{18} * x_2 = x_{18} * x_3 = x_{18} * x_4 \\
&= x_{18} * x_5 = x_{18} * x_6 = x_{18} * x_{10} = x_{18} * x_{11} = x_{18} * x_{12} \\
&= x_{18} * x_{13} = x_{18} * x_{14} = x_{18} * x_{15} = x_{18} * x_{16} = x_{18} * x_{17} \\
&= x_{18} * x_{18} = 0
\end{aligned}
$$

Therefore, we obtain the quasi-associative algebra structure for $\mathfrak{g}_3$ with respect to the basis $S = \{x_1, x_2, x_3, \ldots, x_{18}\}$.

CONCLUSIONS

We proved in Proposition 7 that the Lie algebra $\mathfrak{g}_3$ has quasi-associative algebra structures and we gave their explicit formulas as written in Table 1 above. As discussion, our results can attract other researchers to study quasi-associative algebra structures on the Frobenius Lie algebra $\mathrm{M}_{n,p}(\mathbb{R}) \rtimes \mathfrak{gl}_n(\mathbb{R})$ in the general formulas. Indeed, our result give some adventages to observe the properties of the Frobenius Lie algebra $\mathrm{M}_{n,p}(\mathbb{R}) \rtimes \mathfrak{gl}_n(\mathbb{R})$ particularly for quasi-associative algebra. In formal stage, we state in the following conjecture.

Conjecture 8. *The Lie algebra* $\mathrm{M}_{n,p}(\mathbb{R}) \rtimes \mathfrak{gl}_n(\mathbb{R})$ *(p divides n) with the standard basis* $S = \{x_i\}_{i=1}^{n(n+p)}$ *is the quasi-associative algebra.*

This conjecture is still an open problem to be investigated. The problem is how to find a Frobenius functional for $\mathrm{M}_{n,p}(\mathbb{R}) \rtimes \mathfrak{gl}_n(\mathbb{R})$ and how to relate this Frobenius functional to quasi-associative algebra structures.

AUTHOR CONTRIBUTIONS STATEMENT

HH worked as the main drafter in this research. data collection and instrument design assisted by EK and EC.

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